A Co-reflection of Cubical Sets into Simplicial Sets

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and Liang Ze Wong

HoTT/UF Workshop 2019
CAS Oslo, 13 June

$$\mathbf{sSet} \underset{\mathcal{R}}{\overset{\mathcal{Q}}{\rightleftarrows}} \mathbf{cSet} \quad (\mathcal{Q} \dashv \mathcal{R},\ \mathcal{Q} \text{ an inclusion } \hookrightarrow)$$

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model structures $\rightsquigarrow$ model structures

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Recall the *simplex category* Δ:

- objects are $[n] = \{0 \leq 1 \leq \cdots \leq n\}$
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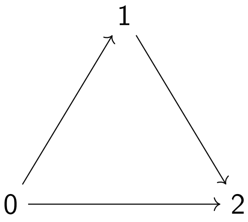

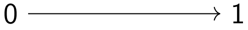

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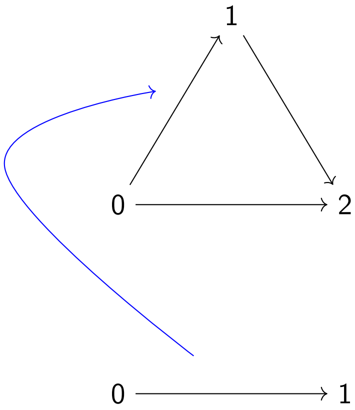

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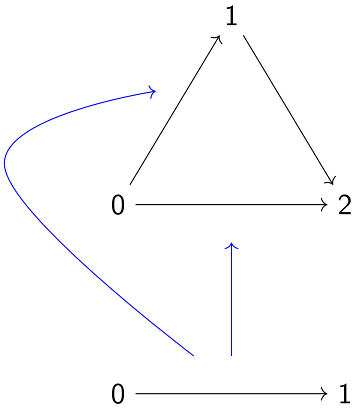

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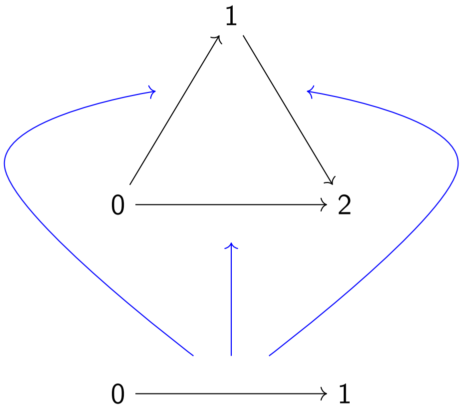

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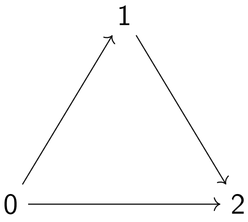

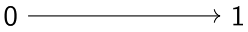

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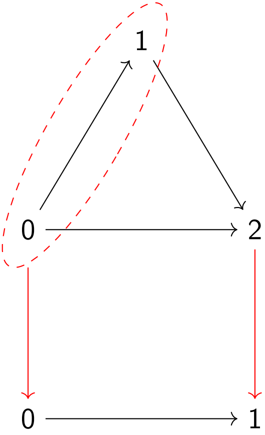

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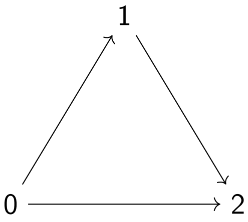

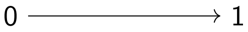

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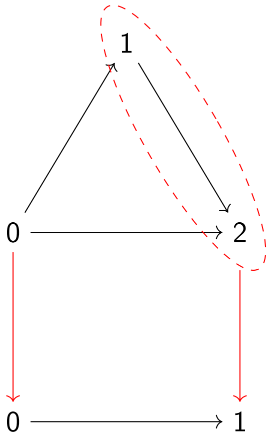

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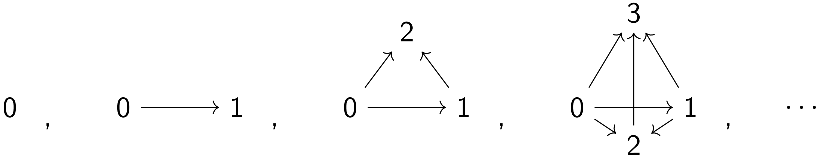

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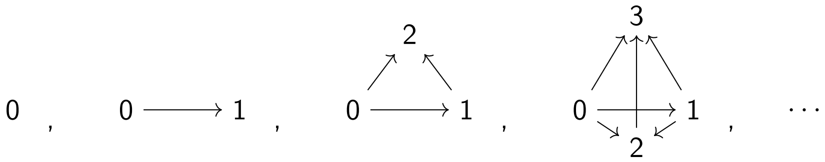

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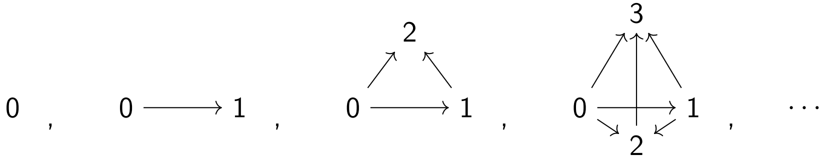

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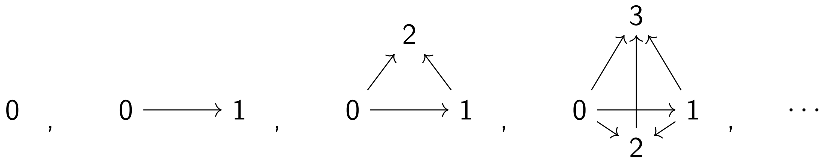

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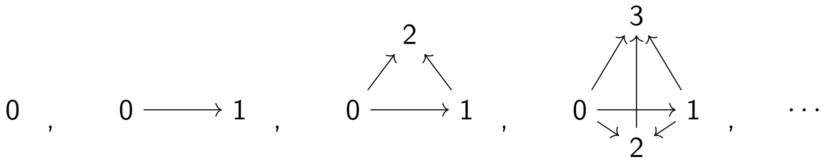

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We also have the *cube category* $\square$:

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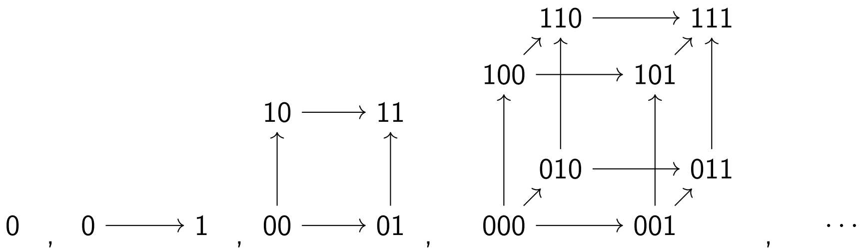

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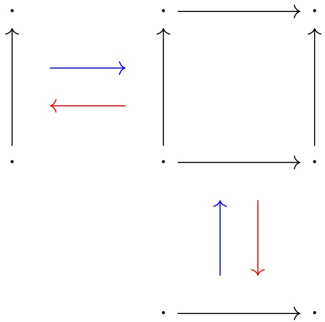

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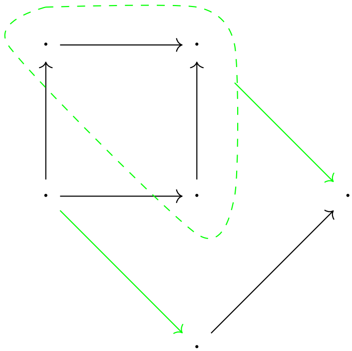

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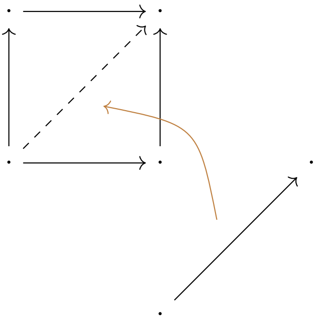

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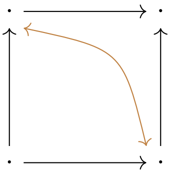

Cubical sets

But for this talk, we will only consider:

- *face* and *degeneracy* maps
- *connections* (max ~~& min~~)
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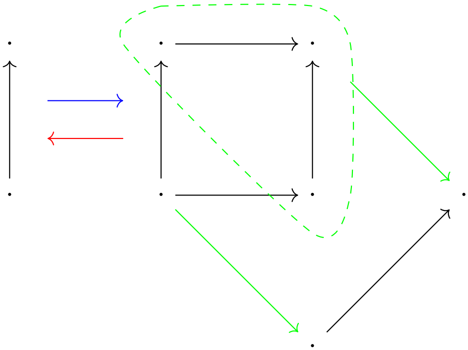


Comparing **cSet** variants

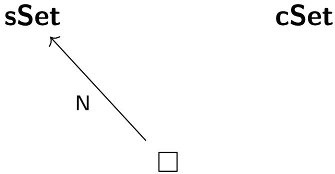
Maps in □	Used in HoTT	(Generalized) Reedy
face-deg-conn		✓
face-deg -symm	BCH[1]	(✓)
face-deg -symm-diag	Cartesian[2]	(✓)
face-deg-conn-symm-diag	CCHM[3]	✗

[1] Bezem-Coquand-Huber 2014

[2] Angiuli-Brunerie-Coquand-Favonia-Harper-Licata 2017

[3] Cohen-Coquand-Huber-Mörtberg 2016

Comparing **cSet** and **sSet**: Triangulation



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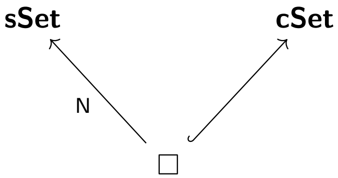

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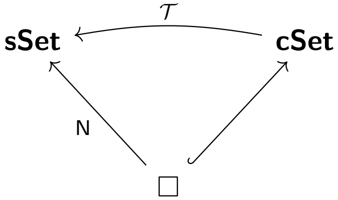

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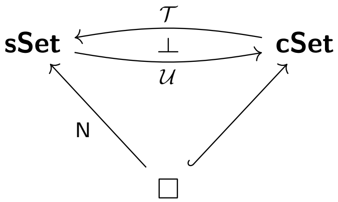

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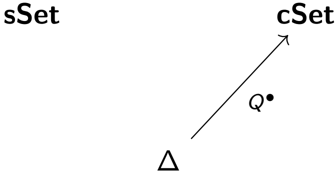

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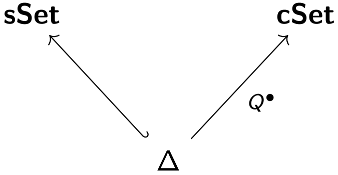

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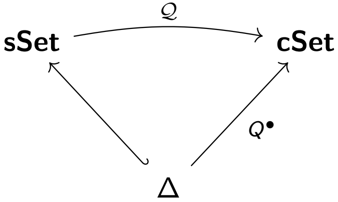

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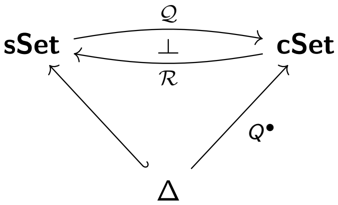

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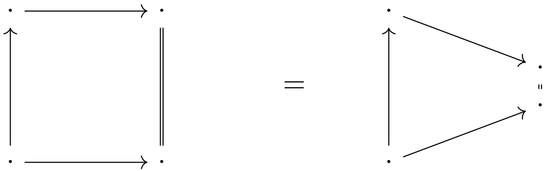

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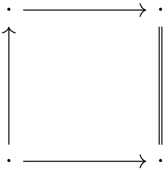

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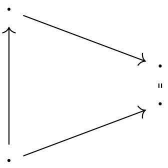

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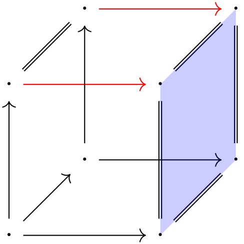

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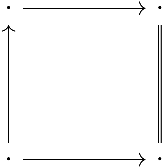

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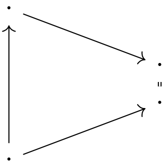

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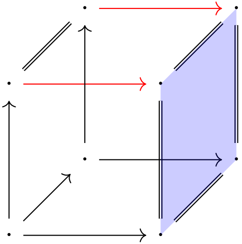

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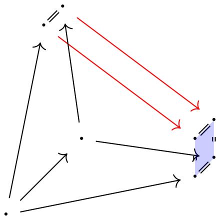

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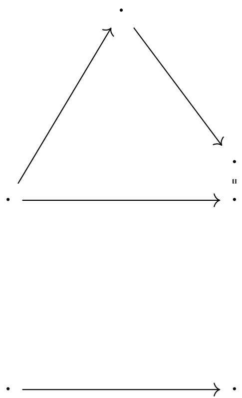

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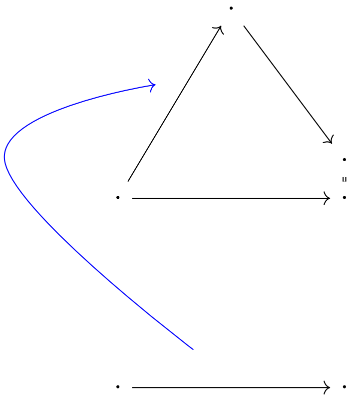

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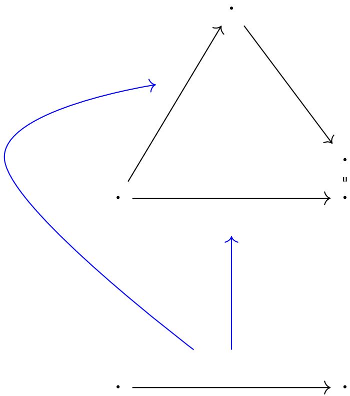

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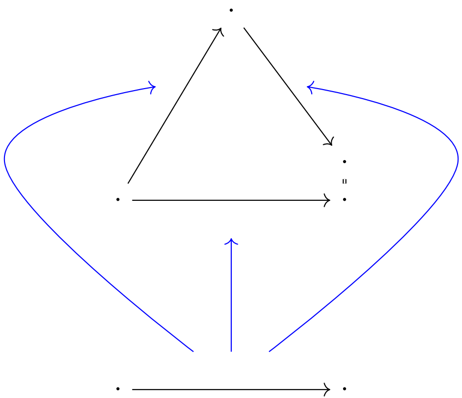

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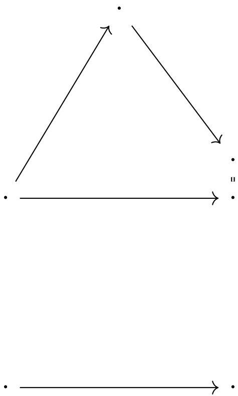

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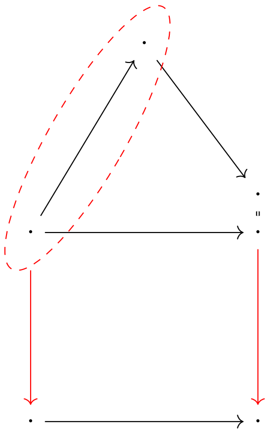

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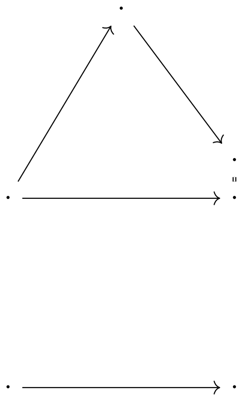

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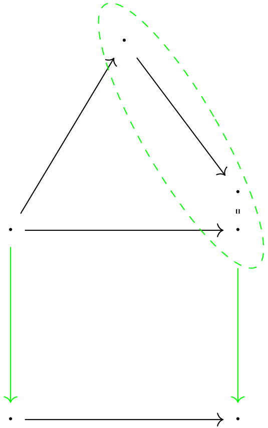

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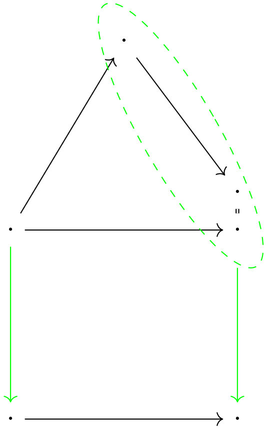

i.e. the Q^n's form a co-simplicial object!

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Proposition (Kapulkin-Lindsey-W., 2019)

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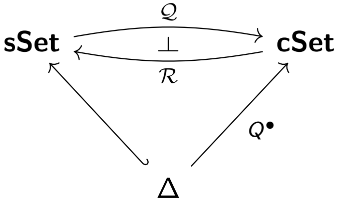

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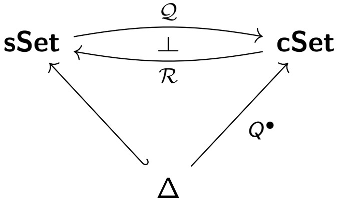

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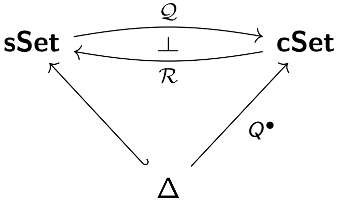

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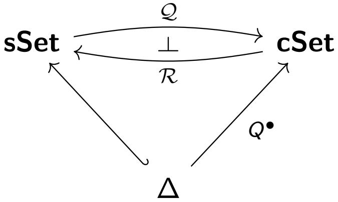

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Theorem (Kapulkin-Lindsey-W., 2019)

$\mathcal{Q} \dashv \mathcal{R}$ defines a co-reflective inclusion of **sSet** *into* **cSet**.

$$\mathbf{sSet} \underset{\mathcal{R}}{\overset{\mathcal{Q}}{\underset{\perp}{\rightleftarrows}}} \mathbf{cSet}$$

(i.e. $\mathcal{Q}$ is fully faithful, and the unit is a natural isomorphism)

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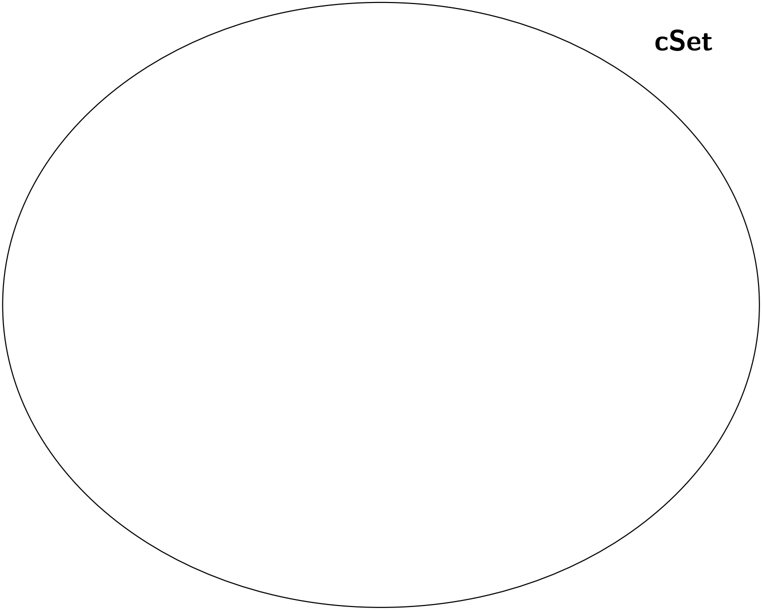

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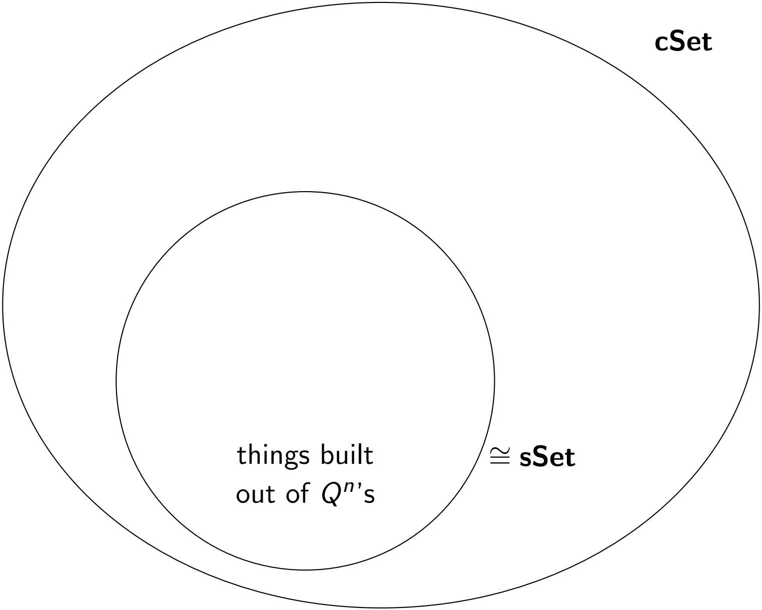

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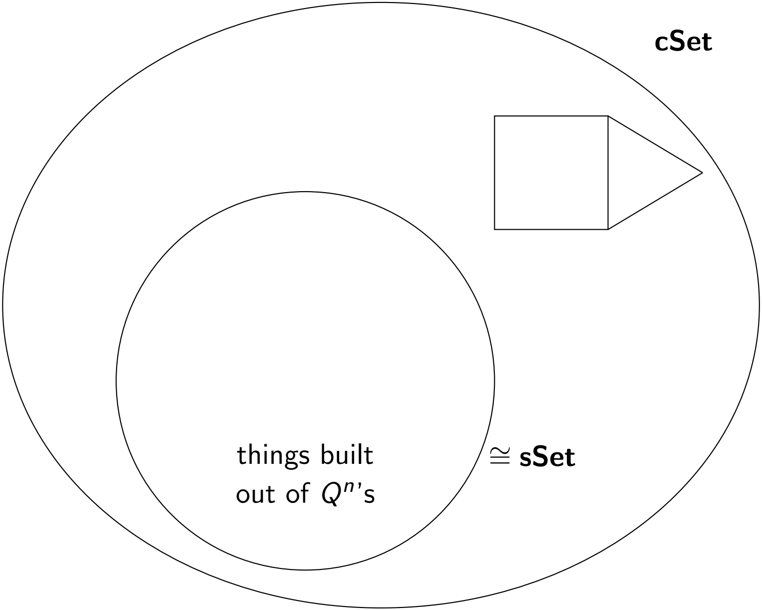

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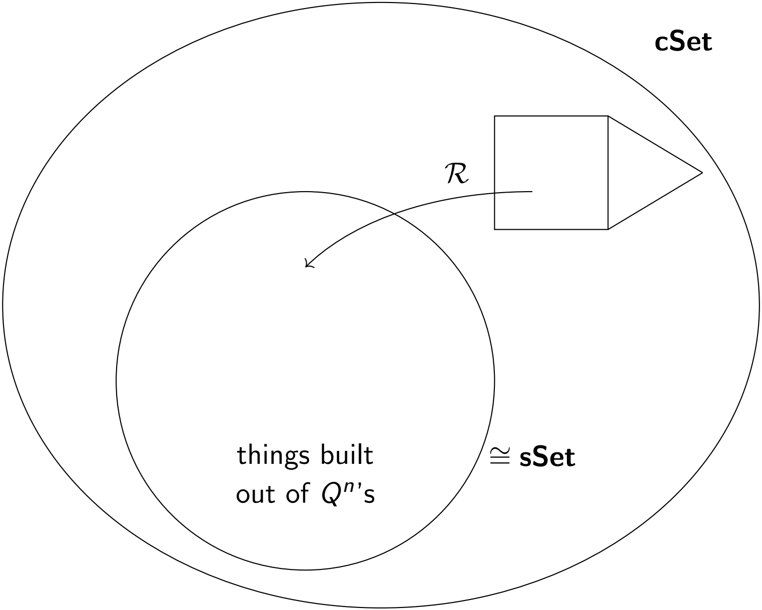

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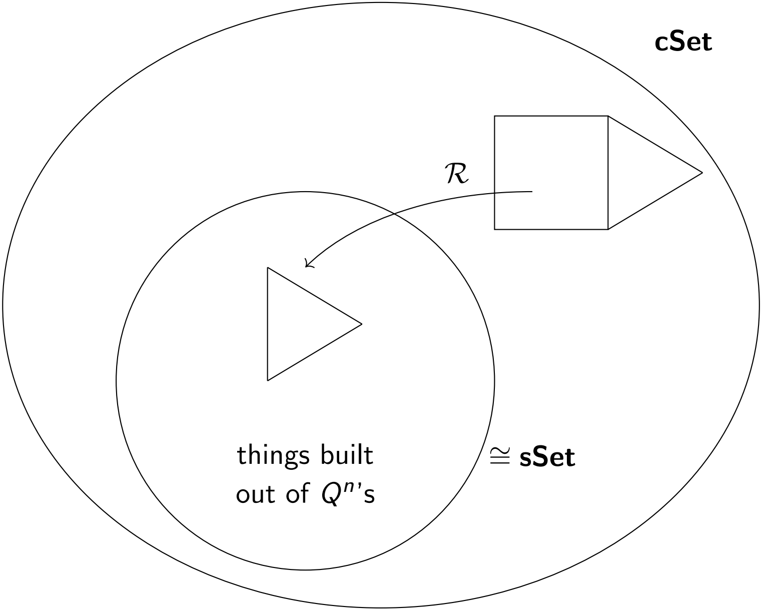

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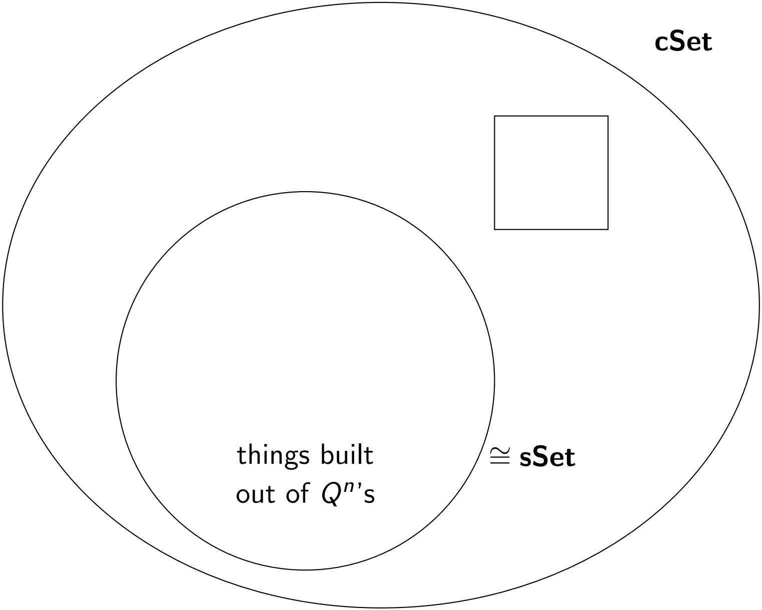

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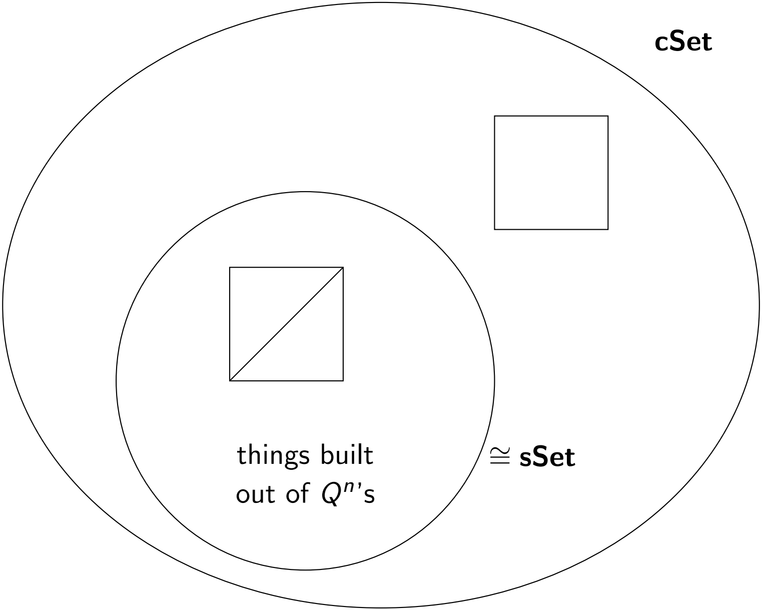

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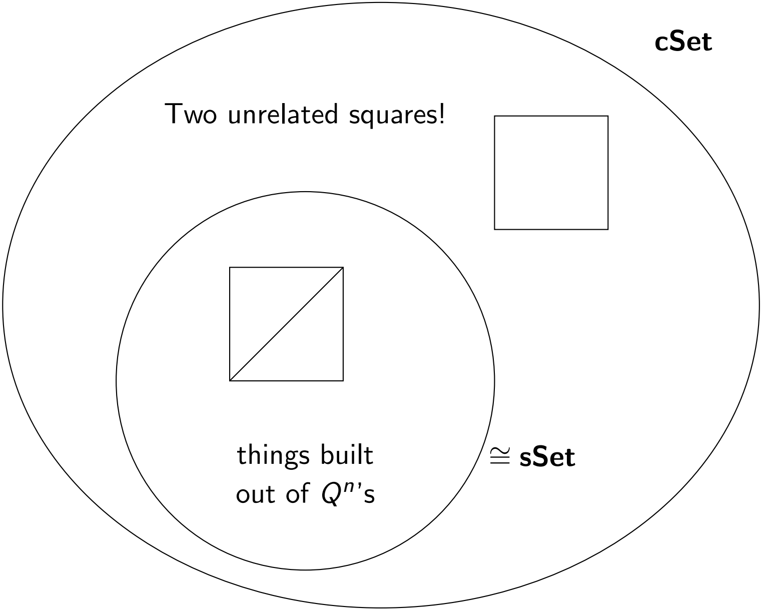
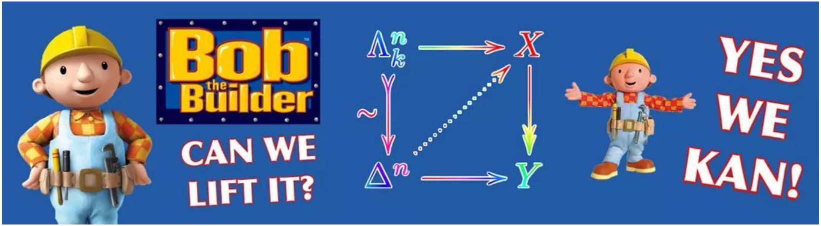


Interlude



Bob the Builder
CAN WE
LIFT IT?
Λ^n_k
X
$\sim$
Δ^n
Y
YES
WE
KAN!

Model Structures

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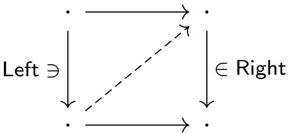

Model Structures

e.g. In the *Quillen model structure* on **sSet**:

$$
\begin{array}{ccc}
\Lambda^n_k & \longrightarrow & X \\
{\scriptstyle \sim}\big\downarrow & \nearrow & \big\downarrow \\
\Delta^n & \longrightarrow & Y
\end{array}
$$

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This allows us to characterize the homotopy category of M as:

$$\operatorname{Ho} M \simeq M_{\text{Cof-Fib}} / \sim$$

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In fact, both of these are *cofibrantly generated* model structures, and the cofibrations are precisely the monomorphisms.

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such that R preserves $\twoheadrightarrow$ and $\xrightarrow{\sim}\!\!\!\!\twoheadrightarrow$.

This is a *Quillen equivalence* if R induces an equivalence:

$$\mathrm{Ho}N \simeq \mathrm{Ho}M$$

Induced Model Structures

Given an adjunction where M is a model category,

$$M \underset{R}{\overset{L}{\underset{\perp}{\rightleftarrows}}} C$$

we may try to *right-induce* a model structure on a bicomplete C by declaring $f \in C$ to be:

- a fibration if Rf is a fibration
- a weak equivalence if Rf is a weak equivalence
- a cofibration if it has the left lifting property (LLP) w.r.t. acyclic fibrations

Induced Model Structures

Proposition (Hess-Kędziorek-Riehl-Shipley '17, Garner-K.-R. '18)

Let M be an accessible model category. An adjunction

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- So just need them to be weak equivalences as well

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Given any cofibranty generated model structure on **sSet** *in which every cofibration is a monomorphism, the adjunction*

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- Implications for type theory?

Thank you!