Distributive Laws[1]

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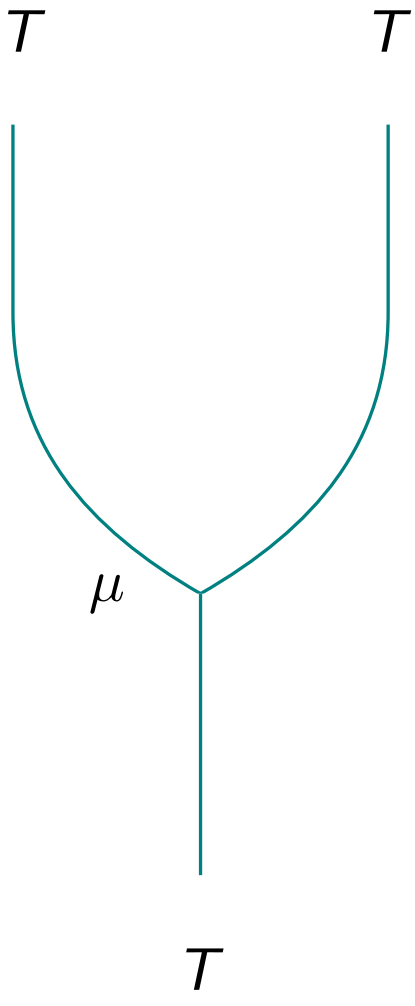
Category Theory 2017

[1] Strings attached!

Outline

1. String diagrams, monads, adjunctions
2. Distributive laws between monads S, T
3. Lifts of monads T to the category of algebras $\mathbf{X}^S$
4. (2) $\Longleftrightarrow$ (3)

String diagrams for monads



$\mathbf{X}$

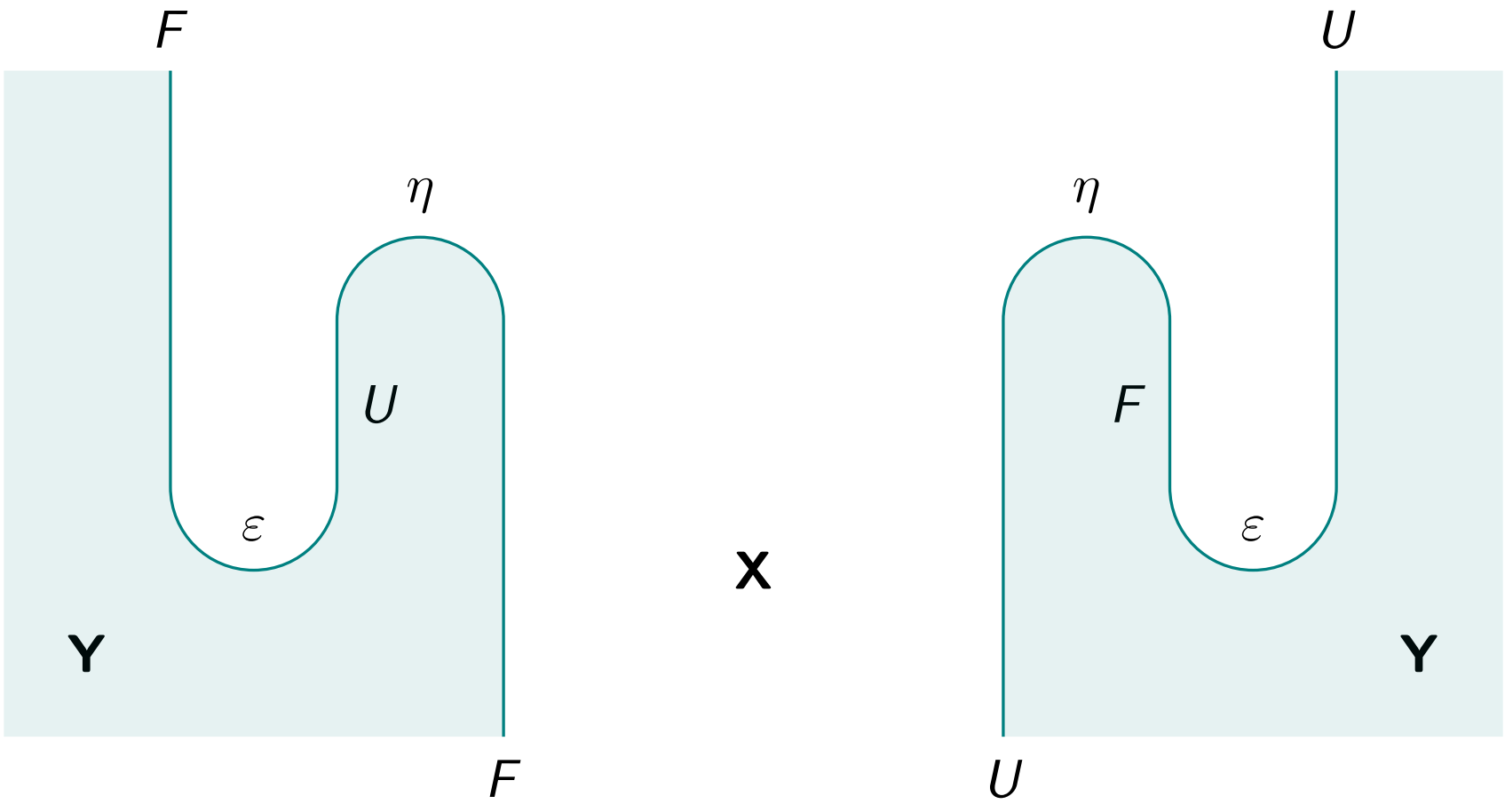
$T : \mathbf{X} \to \mathbf{X}$

$\mu : TT \Rightarrow T$

$\eta : 1_{\mathbf{X}} \Rightarrow T$

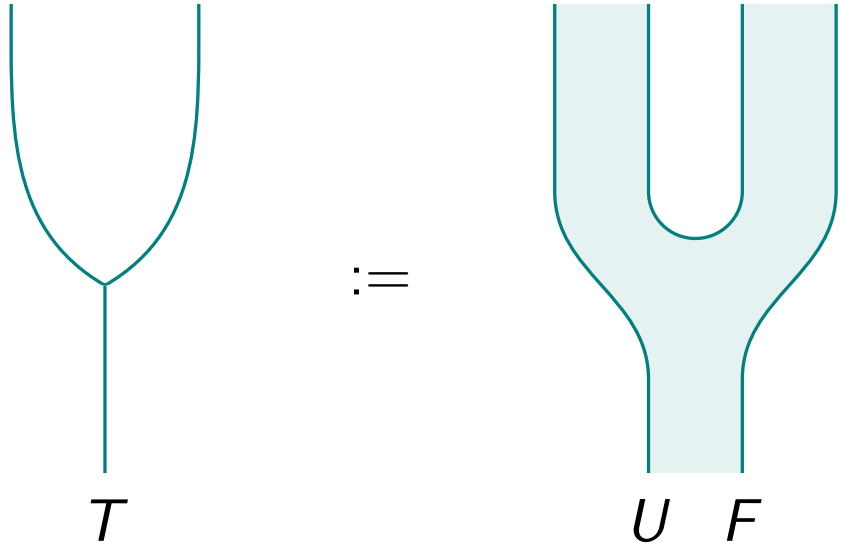
String diagrams for adjunctions

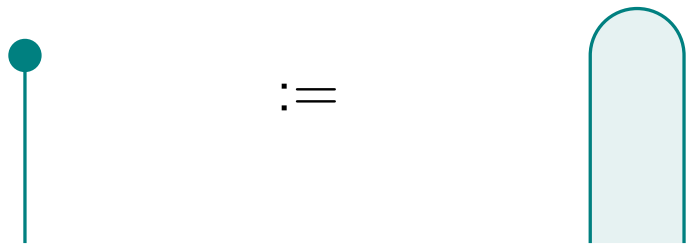
$F : \mathbf{X} \to \mathbf{Y}$ $U : \mathbf{Y} \to \mathbf{X}$ $\eta : 1_{\mathbf{X}} \Rightarrow UF$ $\varepsilon : FU \Rightarrow 1_{\mathbf{Y}}$



Adjunctions give monads give adjunctions

$T := UF$ $\quad \mu_T := U\varepsilon F$ $\quad \eta_T := \eta$





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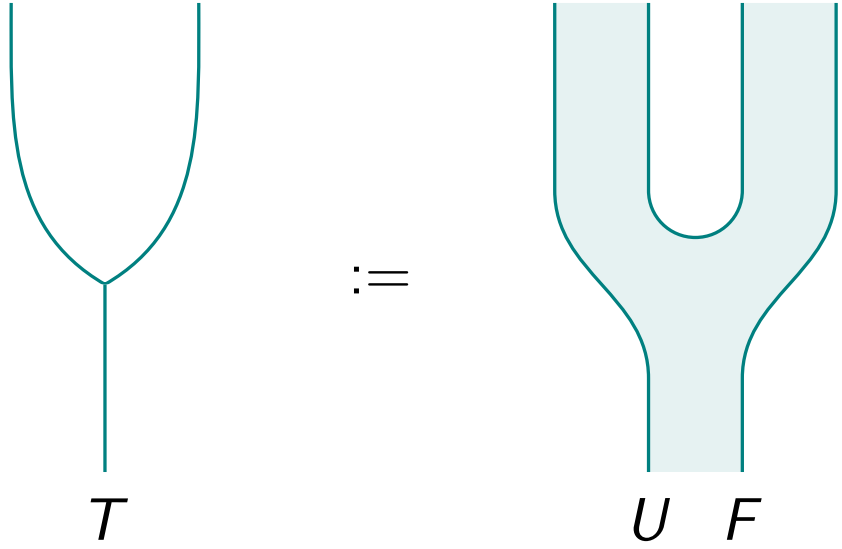
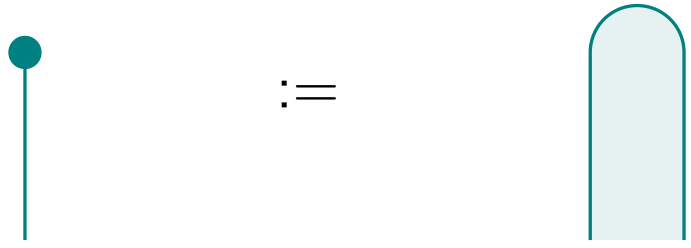


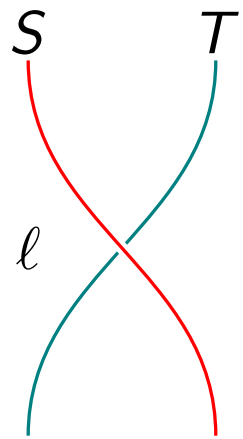


$\mathbf{Y} := \mathbf{X}^T$ $\quad F := F^T$ $\quad U := U^T$

Distributive laws

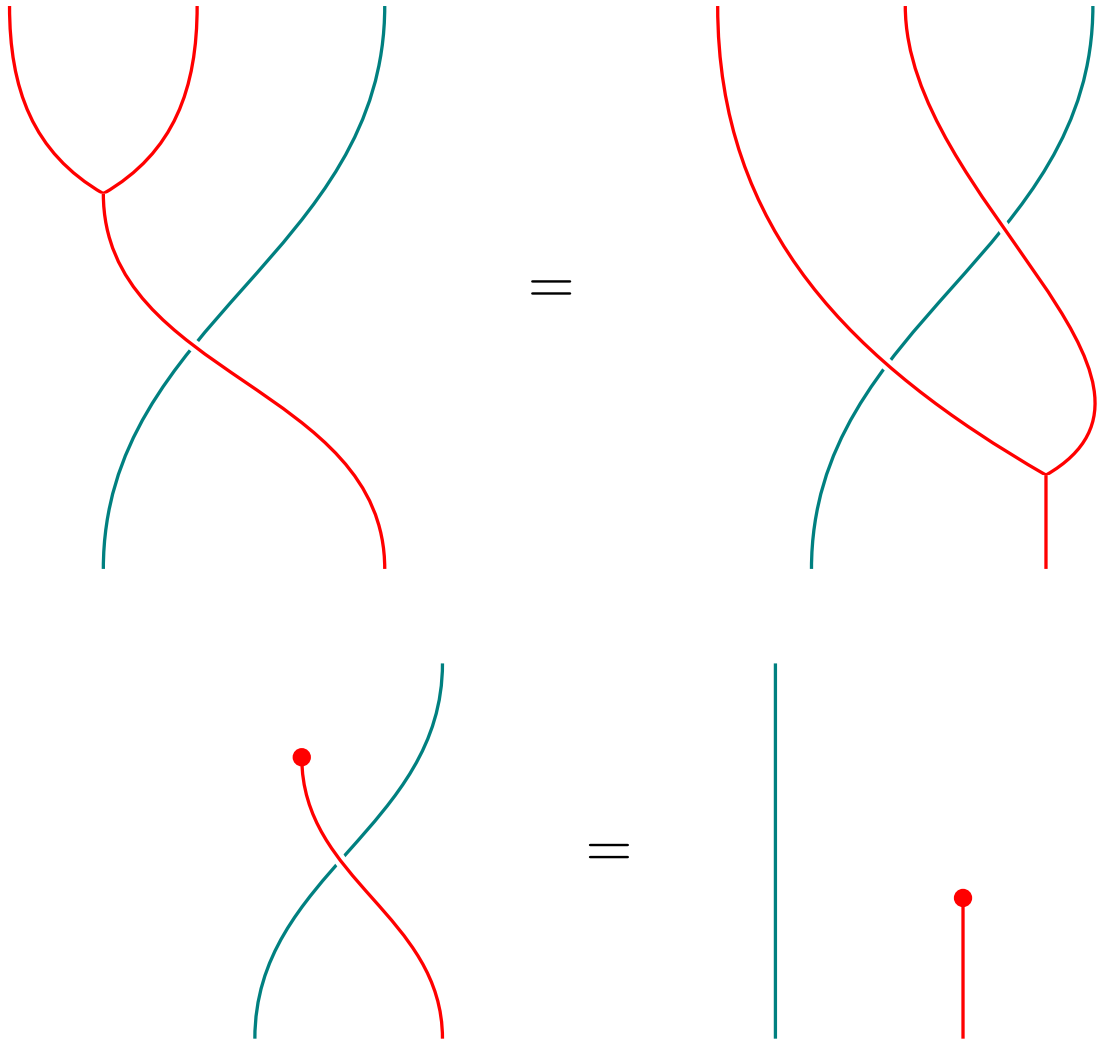
Definition

Let S, T be monads on $\mathbf{X}$. A *distributive law* of S over T is a natural transformation $\ell\colon ST \Rightarrow TS$

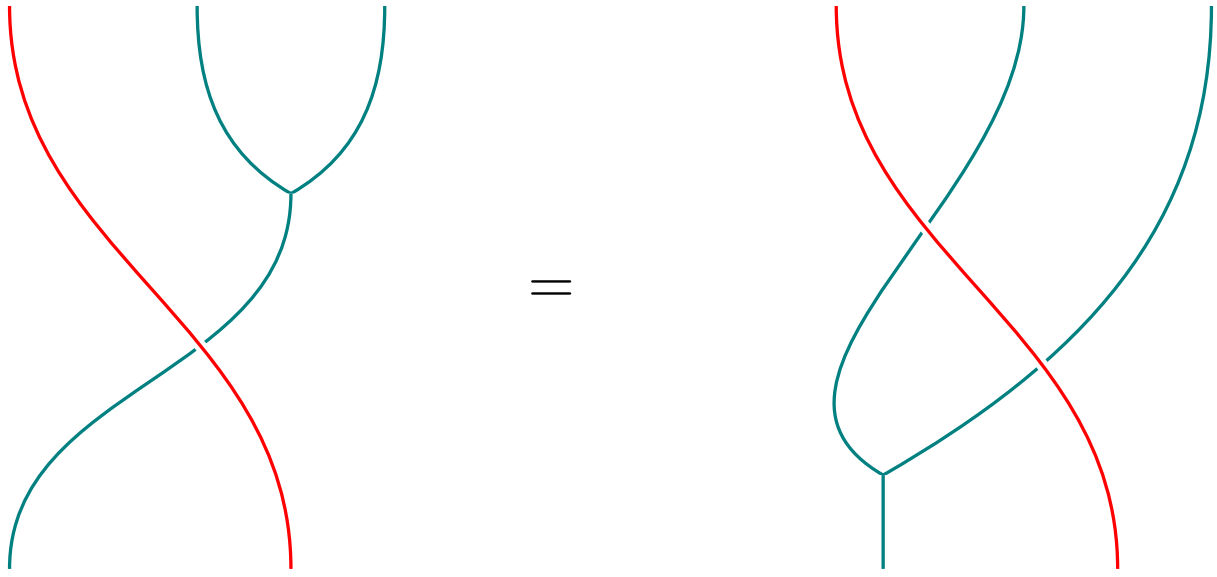


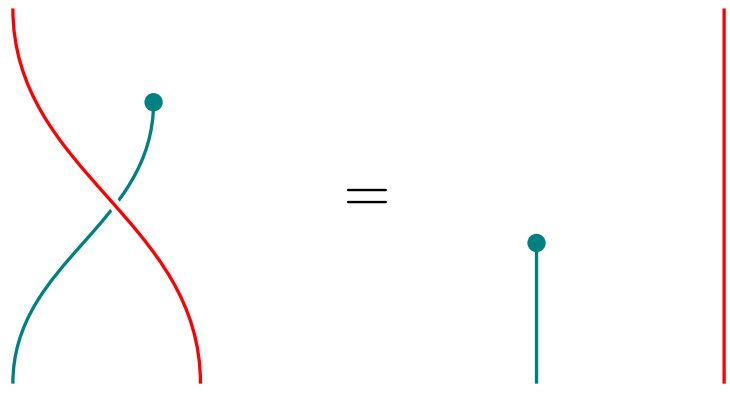
such that ...

Distributive laws



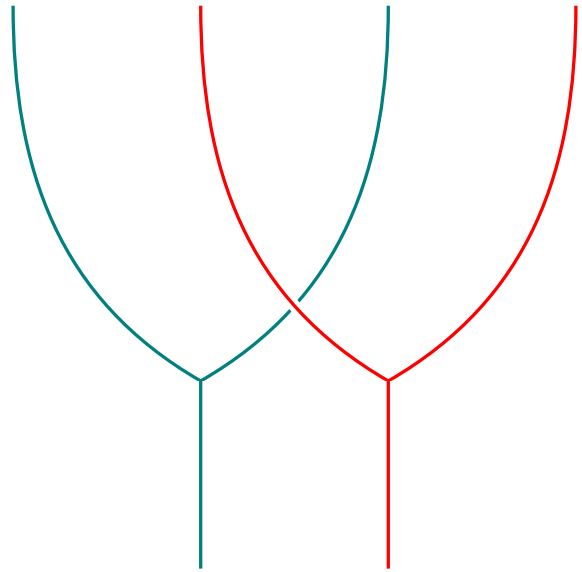
Distributive laws





Distributive laws

A distributive law of S over T makes TS a monad:



But today we'll look at a different characterization.

Lifts of monads

Definition

Let S, T be monads on $\mathbf{X}$. A *lift* of T to $\mathbf{X}^S$ is a monad $(\tilde{T}, \tilde{\mu}^T, \tilde{\eta}^T)$ on $\mathbf{X}^S$ such that

$$U^S \tilde{T} = T U^S \qquad U^S \tilde{\mu}^T = \mu^T U^S \qquad U^S \tilde{\eta}^T = \eta^T U^S$$

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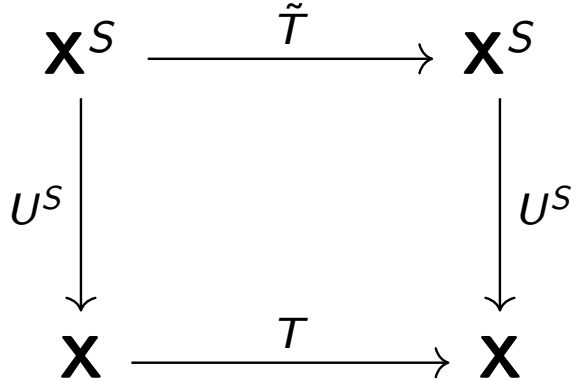

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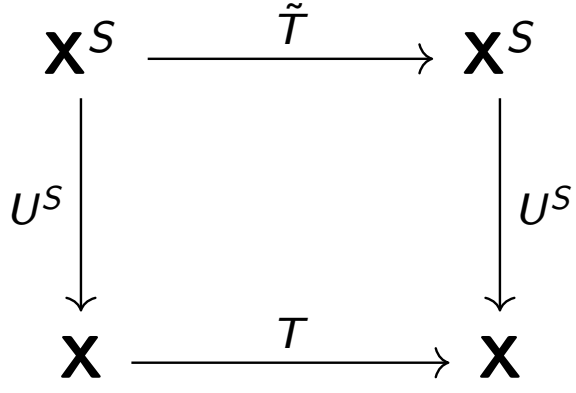

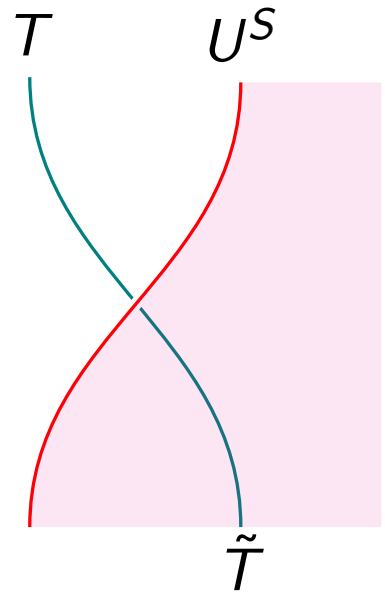

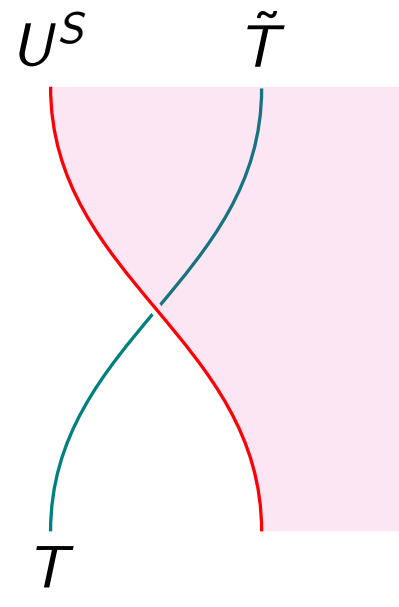

Lifts give distributive laws

Lemma

Let S, T be monads on $\mathbf{X}$ such that T lifts to a monad $\tilde{T}$ on $\mathbf{X}^S$. Then there is a distributive law of S over T.

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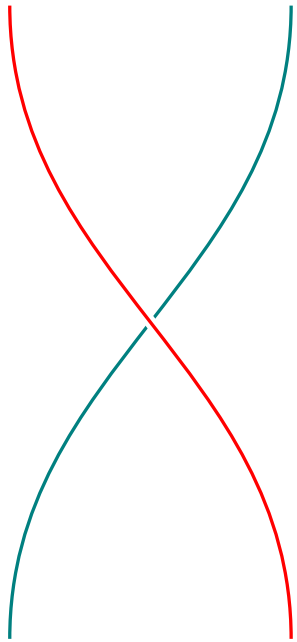

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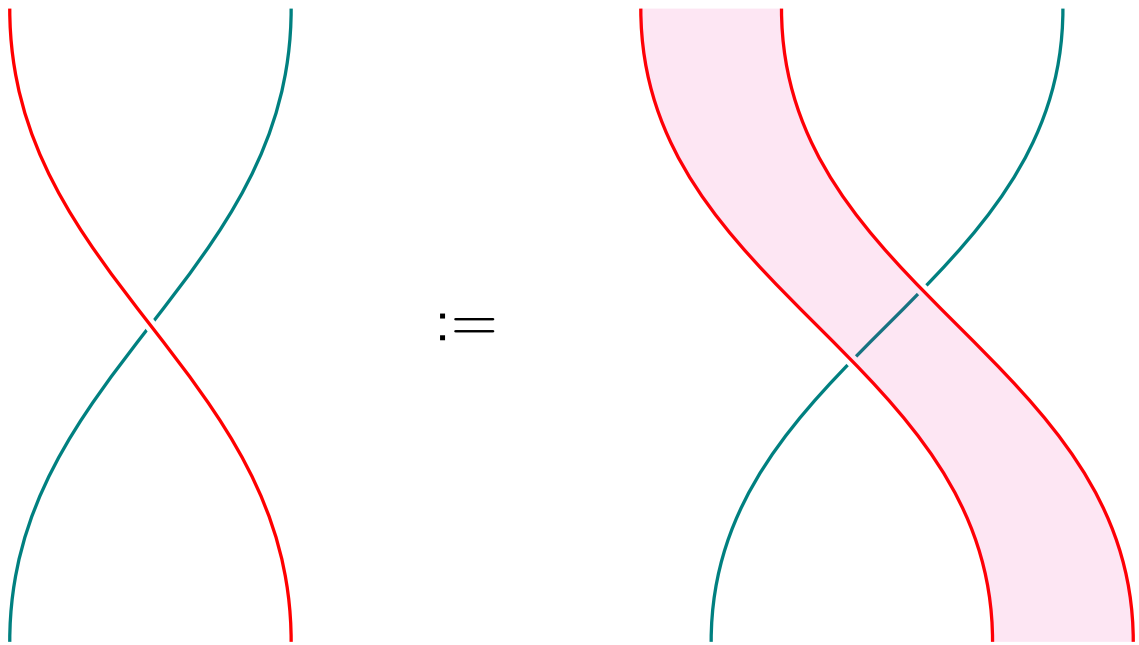

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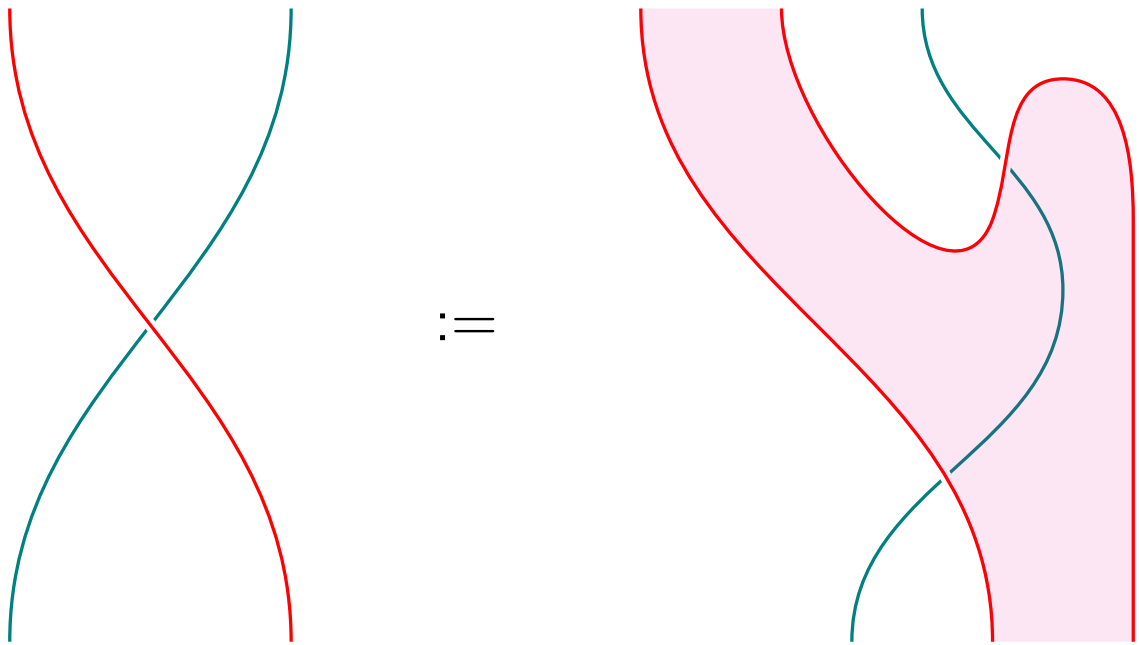

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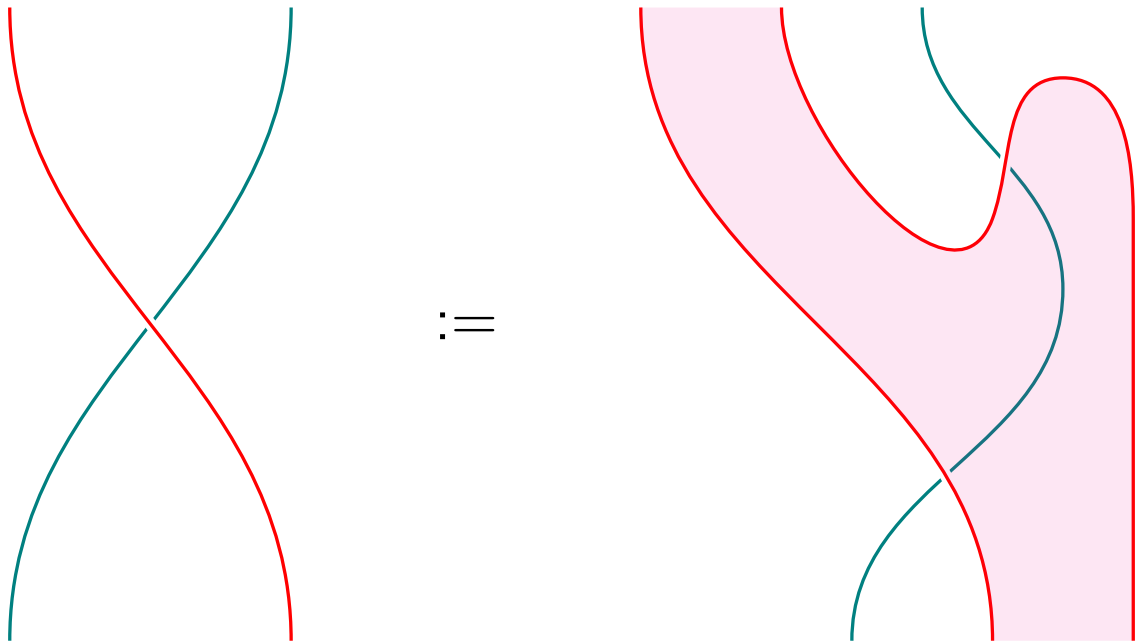


Note: This can be done with lifts over *any* adjunction yielding S.

Distributive laws give lifts

Lemma

Suppose there is a distributive law $\ell\colon ST \Rightarrow TS$ *of* S *over* T*. Then* T *lifts to a monad* $\tilde{T}$ *over* $\mathbf{X}^S$*.*

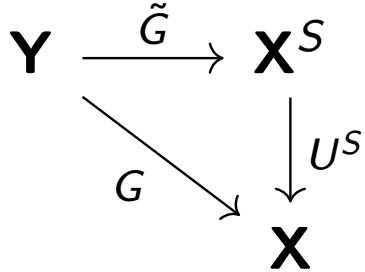
Distributive laws give lifts

Lemma

Suppose there is a distributive law $\ell\colon ST \Rightarrow TS$ of S over T. Then T lifts to a monad $\tilde{T}$ over $\mathbf{X}^S$.

This requires the universal property of $\mathbf{X}^S$:

$$\left\{\text{Functors } \tilde{G} : \mathbf{Y} \to \mathbf{X}^S\right\} \quad \cong \quad \left\{\begin{array}{c}\text{Functors } G : \mathbf{Y} \to \mathbf{X} \\ \text{with } S\text{-action } \sigma : SG \Rightarrow G\end{array}\right\}$$



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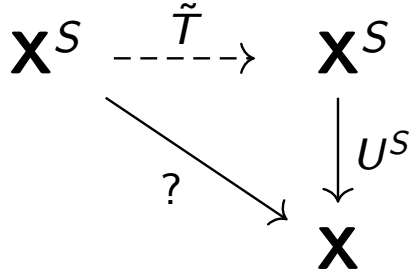

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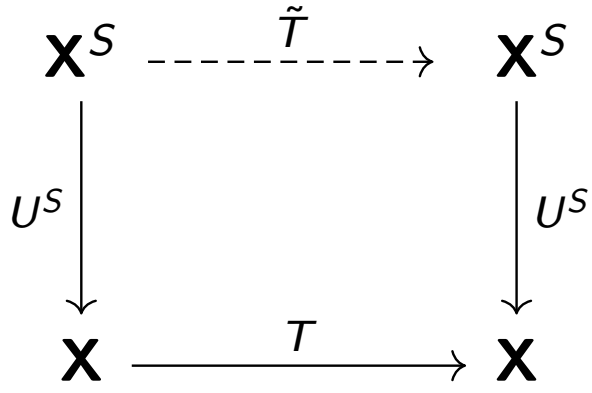

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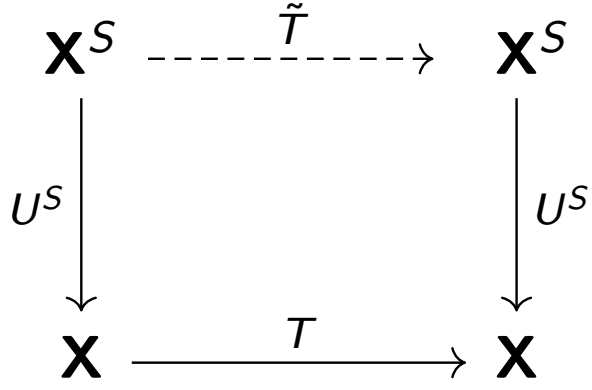

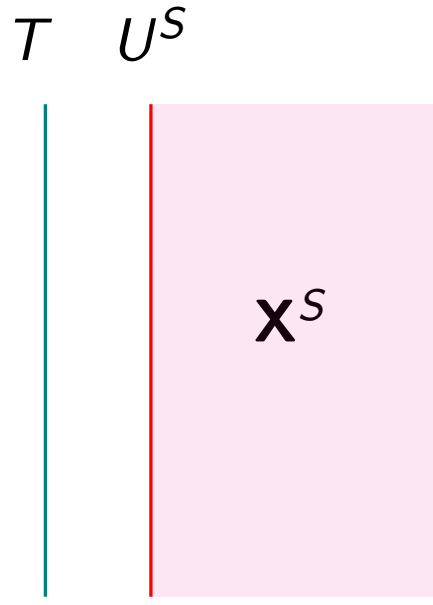

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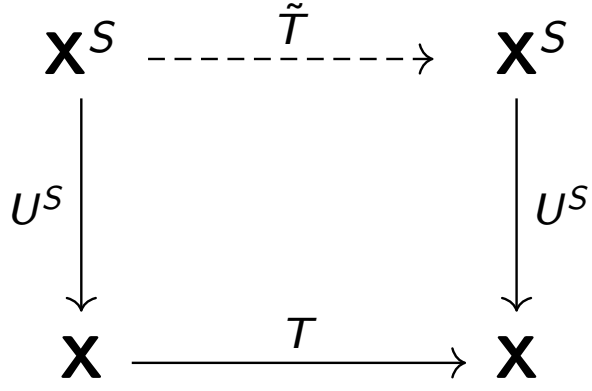
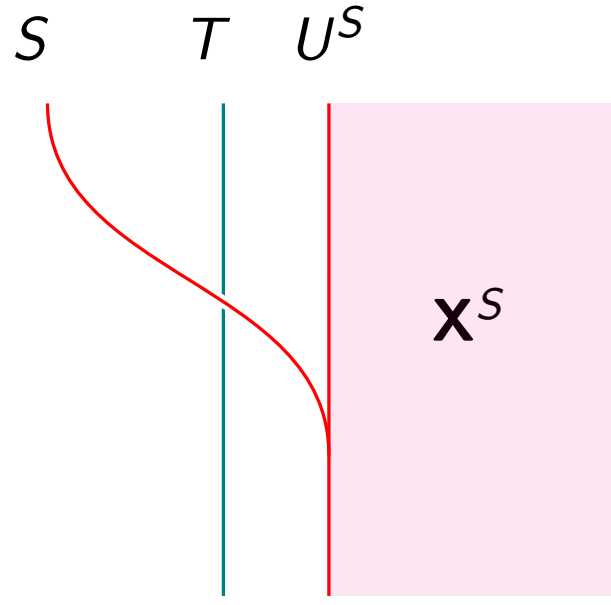
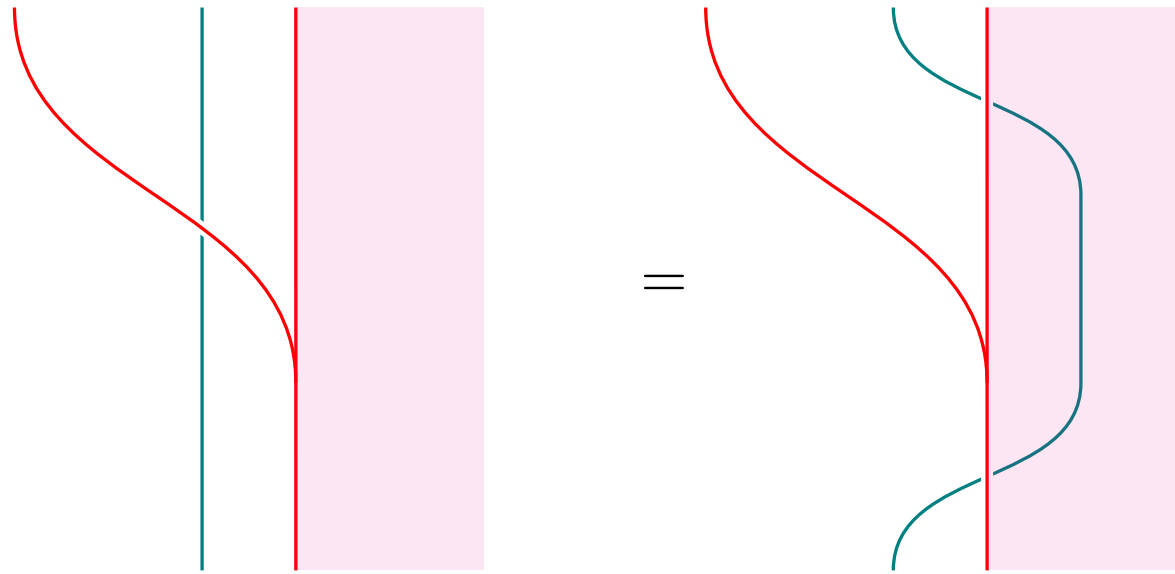
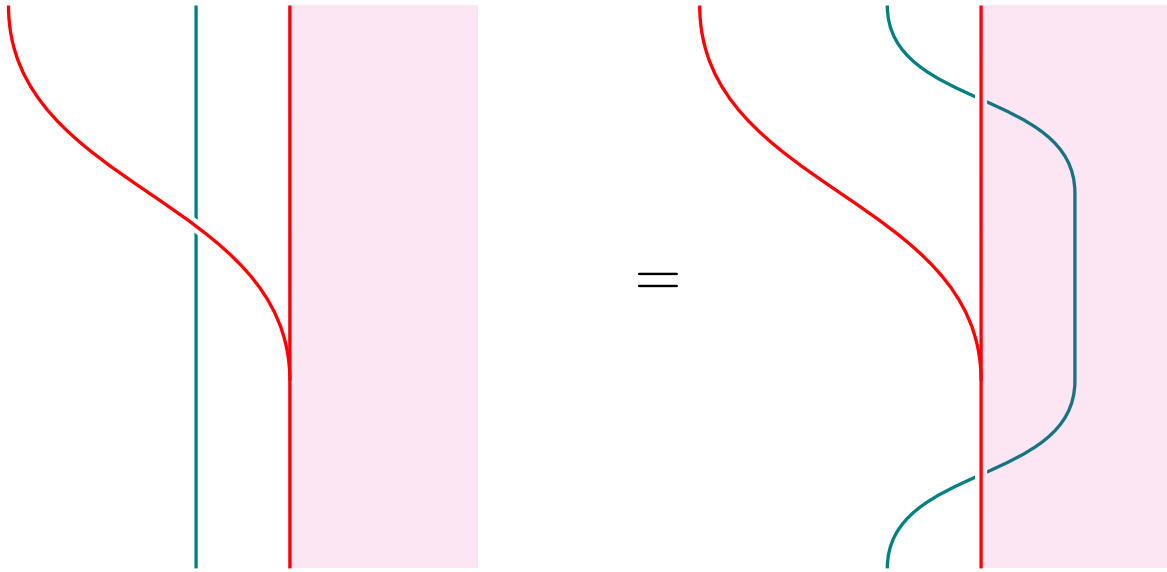








Thank you!

Questions?

References

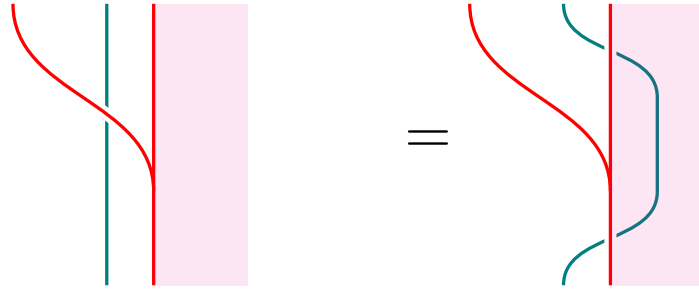
- Jon Beck. *Distributive laws*.
  Seminar on triples and categorical homology theory, 119–140.
  Springer, 1969.

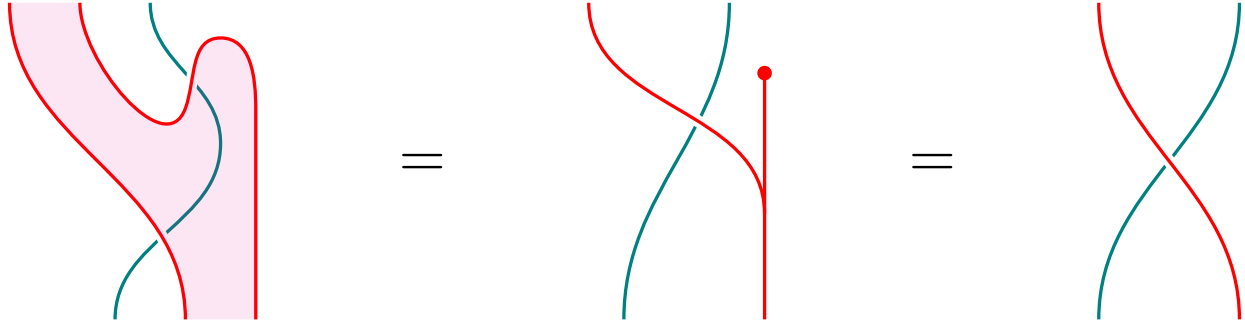
- Eugenia Cheng. *Distributive laws for Lawvere theories.*
  arXiv:1112.3076, 2011.

- Eugenia Cheng. *Distributive laws 1-4* (videos).
  `https://www.youtube.com/playlist?list=PLEC25F0F5AC915192`

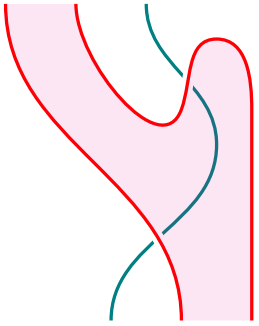
- Ross Street. *The formal theory of monads*.
  Journal of Pure and Applied Algebra, 2(2):149–168, 1972.

Distributive law to lift to distributive law

- Start with a distributive law 
- This gives a lift satisfying 
- Using the lift, define another distributive law. Check that this is the same as the one we started with:



Lift to distributive law to lift

- Starting with a lift, define a distributive law 
- This gives another lift of T, which also precomposes with U^S to yield TU^S.
- To check that they are the same lift, need to check that the induced S-actions on TU^S are the same: