Notes on Beck's *Distributive Laws*

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WARNING!

The notation in this set of notes differs from Beck's paper in the following key ways:

- Beck writes composites in the opposite direction: GF means applying G first, then F. We will use GF to mean F then G .
- 'Triple' = 'monad', 'cotriple' = 'comonad'
- 'Tripleable' = 'monadic', i.e. equivalent to the adjunction involving the category of algebras over monad.

Motivation 1: Multiplication over Addition

Let S be the free monoid monad, T the free abelian group monad.

'Multiplication distributes over addition' means we have a map:

$$STX \to TSX$$
$$\text{e.g. } (a+b)(c+d) \mapsto ac + ad + bc + bd$$

where $X = \{a, b, c, \dots\}$, say.

Further, TS is the free ring monad.

Motivation 2: Tensoring monoids

Let A, B be monoids in a *braided* monoidal category $(\mathcal{V}, \otimes, 1)$.

Then $A \otimes B$ is also a monoid, with multiplication

$$A \otimes B \otimes A \otimes B \xrightarrow{A \otimes tw \otimes B} A \otimes A \otimes B \otimes B \xrightarrow{m_A \otimes m_b} A \otimes B$$

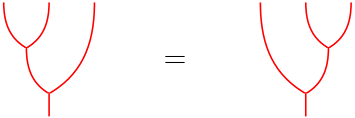
where $tw : B \otimes A \to A \otimes B$ is given by the braiding.

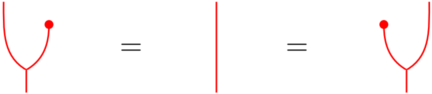
Monads in a 2-category

Fix a 2-category $\mathbf{K}$. A *monad* in $\mathbf{K}$ consists of:

- 0-cell $\mathbf{X}$
- 1-cell $S : \mathbf{X} \to \mathbf{X}$
- 2-cells $\eta^S : 1_{\mathbf{X}} \Rightarrow S$ and $\mu^S : SS \Rightarrow S$

such that



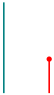


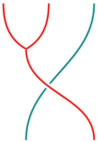
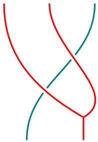
i.e. a monad is a monoid in the monoidal category $(\mathrm{End}(\mathbf{X}), \circ, 1_{\mathbf{X}})$, for some 0-cell $\mathbf{X}$.

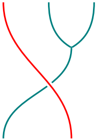
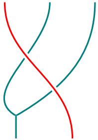
Distributive Law

A *distributive law of* S *over* T is a 2-cell $\ell : ST \Rightarrow TS$

such that:

 =  ;  = 

 = 

 = 

Characterization of Distributive Laws

Characterization

Theorem (Beck 1969, Street 1972, Cheng 2011)

The following are equivalent:

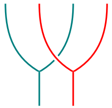
1. Distributive laws $\ell : ST \Rightarrow TS$,
2. Multiplications $m : TSTS \Rightarrow TS$ s.t. $(TS, \eta^T \eta^S, m)$ is monad satisfying the *middle unitary law*, and

$$S \overset{\eta^T S}{\Longrightarrow} TS \overset{T\eta^S}{\Longleftarrow} T$$

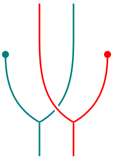
are monad morphisms.
3. *Liftings* of the monad T to a monad $\tilde{T}$ over $\mathbf{X}^S$,
4. *Extensions* of the monad S to a monad $\tilde{S}$ over $\mathbf{X}_T$,
5. Certain elements of $\mathbf{Mnd}(\mathbf{Mnd}(\mathbf{K}))$.

The composite monad

Given $\ell : ST \Rightarrow TS$, define $m : TSTS \Rightarrow TS$ to be

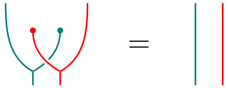


To get back ℓ, do:

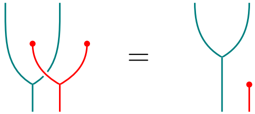


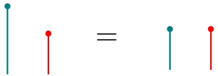
The composite monad

The *middle unitary law* holds:



and $T\eta^S : T \Rightarrow TS$ is a monad morphism:

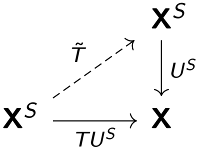




Similarly, $\eta^T S : S \Rightarrow TS$ is a monad morphism.

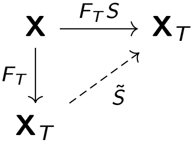
Liftings and Extensions

A *lift* of T to the EM object $\mathbf{X}^S$ is a monad $\tilde{T}$:



$+$ compatibility equations

An *extension* of S to the Kleisli object $\mathbf{X}_T$ is a monad $\tilde{S}$:



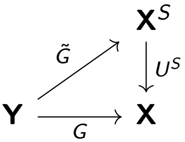
$+$ compatibility equations

Kleisli objects in $\mathbf{K}$ are EM objects in $\mathbf{K}^{op}$, so proofs for liftings hold for extensions too, by duality.

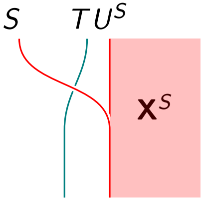
Liftings and Extensions

Universal property[1] of $\mathbf{X}^S$:

$$\left\{ \text{Functors } \tilde{G} : \mathbf{Y} \to \mathbf{X}^S \right\} \quad \cong \quad \left\{ \begin{array}{c} \text{Functors } G : \mathbf{Y} \to \mathbf{X} \\ \text{with } S\text{-action } \sigma : SG \Rightarrow G \end{array} \right\}$$



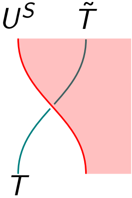
Let $\mathbf{Y} = \mathbf{X}^S$, $G = TU^S$, $\tilde{T} = \tilde{G}$. Need S-action $STU^S \Rightarrow TU^S$.
Given by distributive law and canonical action of S on U^S:



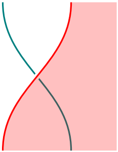
[1] In fact, this is an equivalence of *categories*

Liftings and Extensions

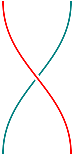
Conversely, a lifting $\tilde{T}$ means we have invertible 2-cells:



with inverse



Lets us define a distributive law:



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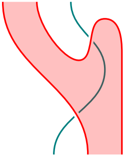
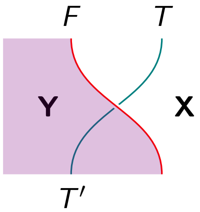


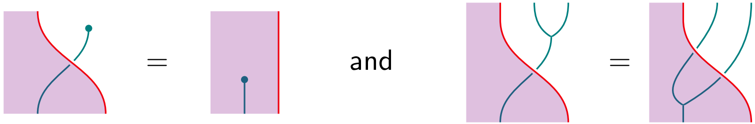
This works for lifts over any adjunction that gives S!

Monads in $\mathbf{Mnd}(\mathbf{K})$

Let $(\mathbf{X}, T), (\mathbf{Y}, T')$ be monads in $\mathbf{K}$.
A *monad opfunctor* $(F, \phi) : (\mathbf{X}, T) \to (\mathbf{Y}, T')$ consists of
$F : \mathbf{X} \to \mathbf{Y}$ and $\phi : FT \Rightarrow T'F$



such that



Monads in $\mathbf{Mnd}(\mathbf{K})$

A *monad functor transformation* is a 2-cell $\sigma : F \Rightarrow F'$ such that

These form a 2-category $\mathbf{Mnd}^*(\mathbf{K})$.

When $\mathbf{X} = \mathbf{Y}$, $T = T'$, if $(F, \phi) : (\mathbf{X}, T) \to (\mathbf{X}, T)$ is a monad, then F is a monad on $\mathbf{X}$ and ϕ is a distributive law of F over T!

$$\text{i.e.}^2\ \mathbf{Dist}(\mathbf{K}) \cong \mathbf{Mnd}^*(\mathbf{Mnd}^*(\mathbf{K}))$$

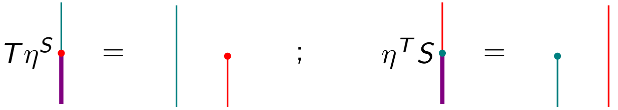
Also, $\mathbf{Mnd}^*$ is a monad!

[2] Can define morphisms between distributive laws such that this is true!

Algebras over TS

Actions of T, S and TS

From before, have monad morphisms[3]: $T \xrightarrow{T\eta^S} TS \xleftarrow{\eta^T S} S$



These induce T- and S-actions on U^{TS}, via the action of TS:

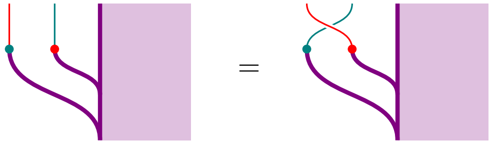
In some sense, any TS-action is 'captured' by these two actions!

[3]Monad opfunctors with $F = 1_{\mathbf{X}}$.

Actions of T, S and TS

Combining T- and S-actions on U^{TS} gives canonical action of TS:

Can then show that the S-action 'distributes over' the T-action:



Algebras over TS

Let ℓ be a distributive law of S over T. From the characterization theorem, we get monads TS on $\mathbf{X}$, $\tilde{T}$ on $\mathbf{X}^S$ and $\tilde{S}$ on $\mathbf{X}_T$.

Theorem (Beck 1969, Cheng 2011)

The category of algebras of TS coincides with that of $\tilde{T}$.

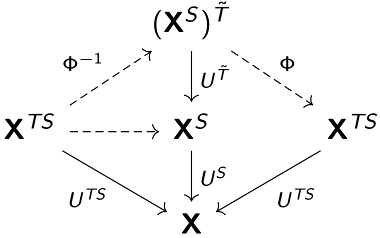
$$\mathbf{X}^{TS} \cong (\mathbf{X}^S)^{\tilde{T}}$$

Dually, the Kleisli category of TS coincides with that of $\tilde{S}$.

$$\mathbf{X}_{TS} \cong (\mathbf{X}_T)_{\tilde{S}}$$

Algebras over TS

Construct $\Phi : \mathbf{X}^{TS} \to (\mathbf{X}^S)^{\tilde{T}}$ and inverse Φ^{-1} as lifts arising from universal properties of $\mathbf{X}^S, (\mathbf{X}^S)^{\tilde{T}}, \mathbf{X}^{TS}$:

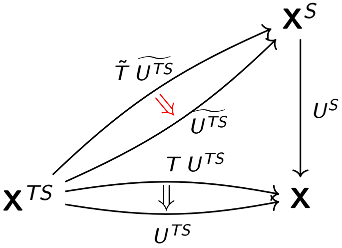


To get Φ^{-1}, need S-action on U^{TS} and $\tilde{T}$-action on lift of U^{TS}.
To get Φ, need TS action on $U^S U^{\tilde{T}}$.

Algebras over TS

We already have T- and S-actions on U^{TS}.

S-action gives a lift $\widetilde{U^{TS}} : \mathbf{X}^{TS} \to \mathbf{X}^S$ of U^{TS}.
To get $\tilde{T}$-action on $\widetilde{U^{TS}}$, lift[4] T-action on U^{TS} :



So we have

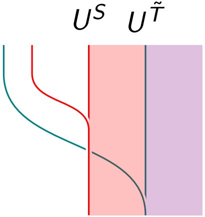
$$\Phi^{-1} : \mathbf{X}^{TS} \to (\mathbf{X}^S)^{\tilde{T}}$$

[4]Need T-action to be an S-alg. morphism, but this follows from distributivity of S-action over T-action.

Algebras over TS

To get $\Phi : (\mathbf{X}^S)^{\tilde{T}} \to \mathbf{X}^{TS}$, need TS-action on $U^S U^{\tilde{T}}$.

Use canonical actions of S on U^S and $\tilde{T}$ on $U^{\tilde{T}}$:



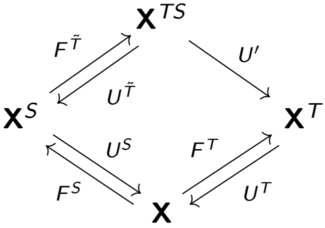
So $\mathbf{X}^{TS} \cong (\mathbf{X}^S)^{\tilde{T}}$, and in fact,

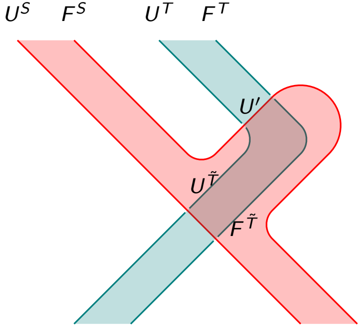
$$U^{TS} F^{TS} = TS = U^S U^{\tilde{T}} F^{\tilde{T}} F^S$$

Distributivity of Adjoints

Distributivity of adjoints

A distributive law gives rise to a 'distributive square':



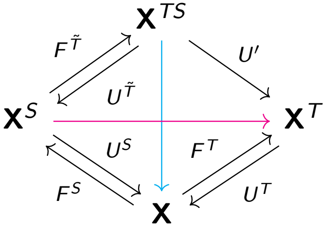


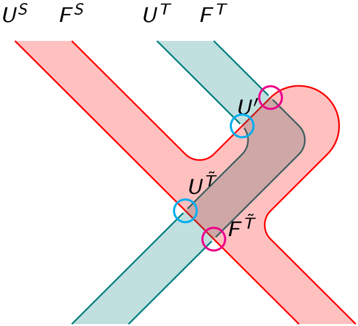
where U' is induced by the T-action on U^{TS}.
If certain coequalizers exist, U' has a left adjoint[5].

[5] Think of U' as 'restriction of scalars', and adjoint as 'extension of scalars'

Distributivity of adjoints

Both composites $\mathbf{X}^{TS} \to \mathbf{X}$ are the same: $U^S U^{\tilde{T}} = U^T U'$.
Both composites $\mathbf{X}^S \to \mathbf{X}^T$ are the same: $U' F^{\tilde{T}} = F^T U^S$.



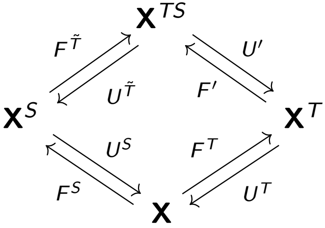


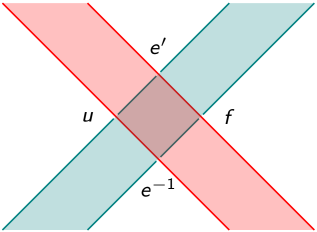
This is a *distributive adjoint situation*, and there is an adjunction:

$$\mathbf{Dist\ Adj\ X} \underset{\text{Sem}}{\overset{\text{Struc}}{\rightleftarrows}} (\mathbf{Dist\ X})^{op} \quad \perp$$

Distributivity of adjoints

If U' has an adjoint F':





To get distributive law:
Need isomorphisms u, f that are 'dual' to each other. These give rise to e, e'.
But e goes in the 'wrong' direction, so need e to be an isomorphism too, to get e^{-1}.

Thank you!

Questions?

References

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