Enriched Fibrations and Comodule Categories

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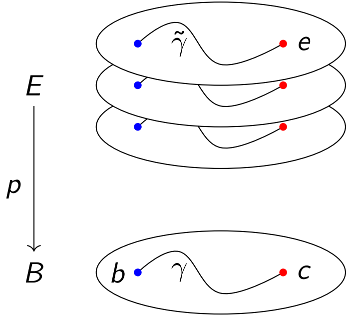
8 Mar 2018

Covering maps



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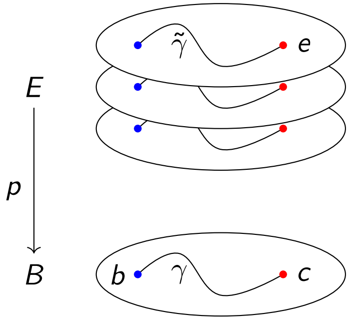

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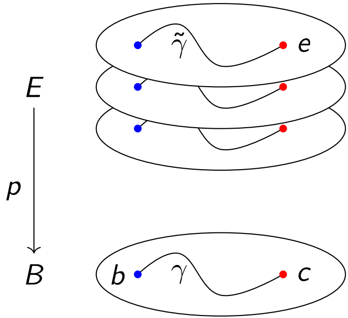

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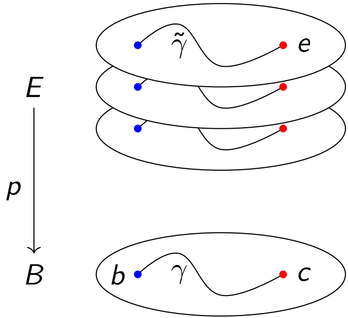

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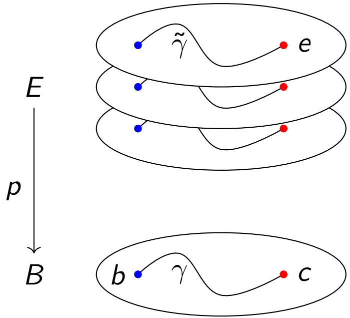

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- $\pi_1(B)$ acts on fibers

$$\pi_1(B)^{op} \to \mathbf{Set}$$

Étale spaces and (pre)sheaves

$$\begin{tikzcd} E \arrow[d, "p"] \\ X \end{tikzcd} \qquad \leftrightarrow \qquad \mathrm{Open}(X)^{op} \to \mathbf{Set}$$

Étale spaces over X Sheaves on X

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Étale spaces over X Sheaves on X

In both these situations, there is a duality between spaces varying nicely over X and sets indexed 'by X'.

Fibrations and indexed categories

Theorem (Grothendieck, 1964)

Let B be a category. There is a 2-equivalence

$$\mathbf{Fib}(B) \cong 2\text{-}\mathbf{Fun}(B^{op}, \mathbf{Cat})$$

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Goal: An enriched version of this result.

Outline

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Grothendieck fibrations

Definition (Grothendieck 1959)

A functor $p \colon E \to B$ is a **fibration** if for every $e \in E$ and $f : b \to pe$ in B, there exists a cartesian lift of f with codomain e.

$$\begin{tikzcd} & e \arrow[d, dotted, no head] & E \arrow[d, "p"] \\ b \arrow[r, "f"'] & pe & B \end{tikzcd}$$

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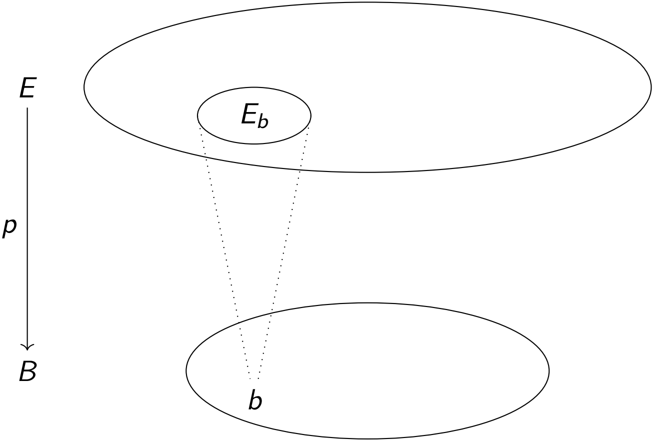
So instead of a *category* with pullbacks, a fibration is like a *functor* 'with pullbacks'.

Fibers of a fibration

The **fiber** E_b consists of objects over b and morphisms over 1_b.



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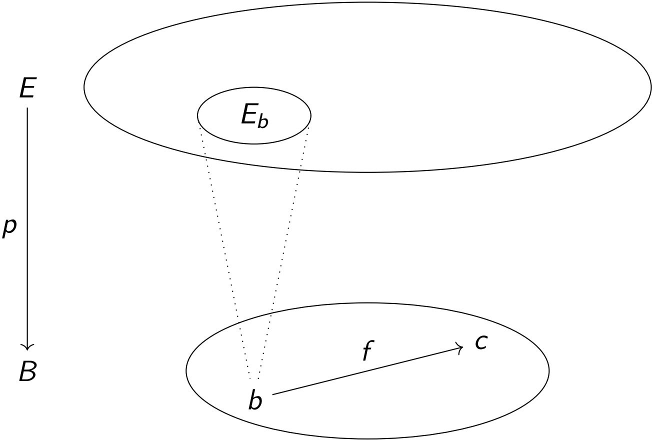

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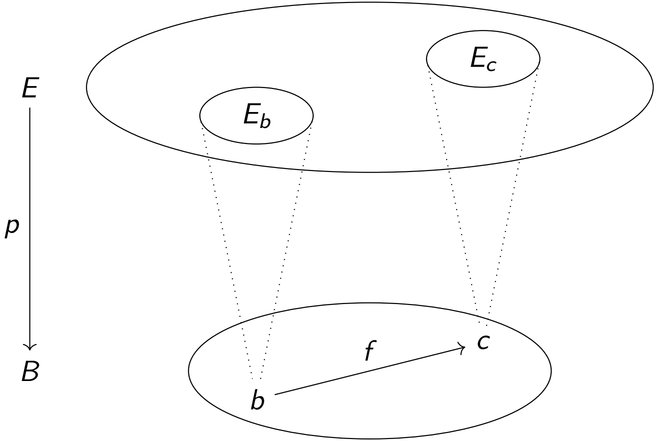

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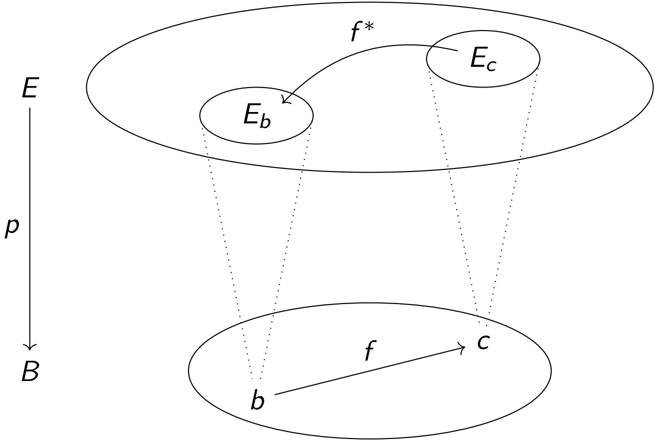
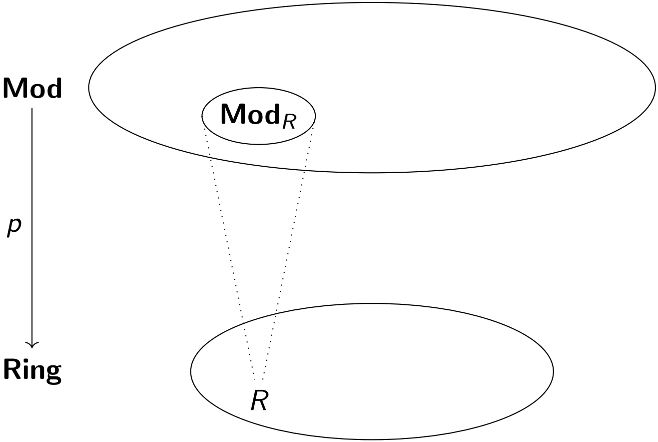


Example 2: the global module category

Mod is the category of pairs (M, R) where $M \in \mathbf{Mod}_R$.



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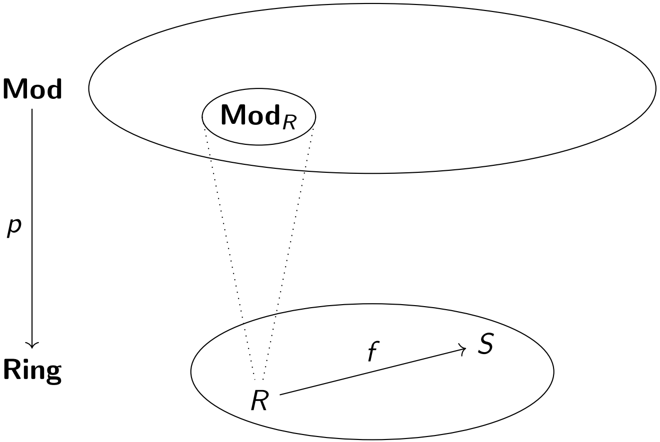

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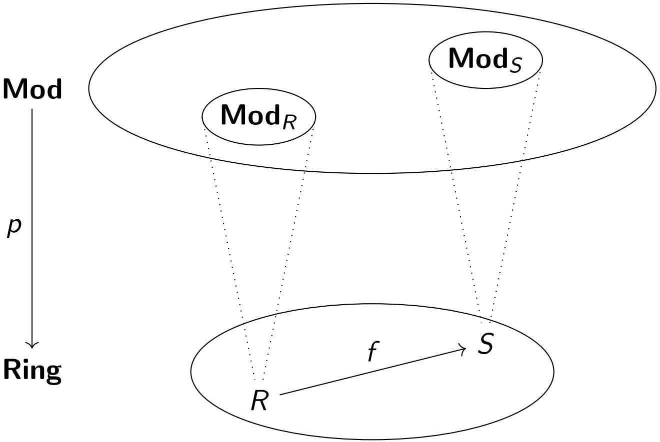

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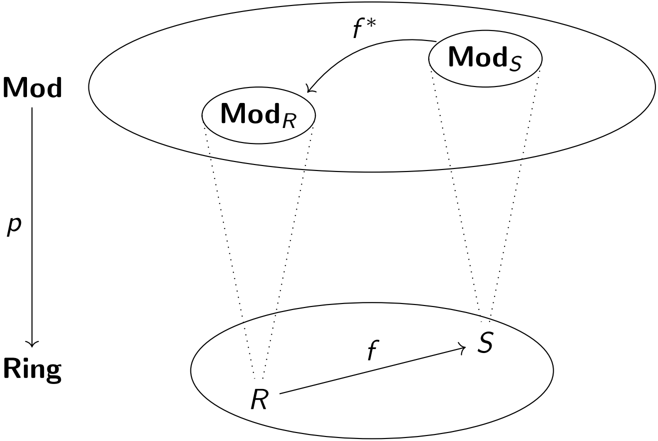

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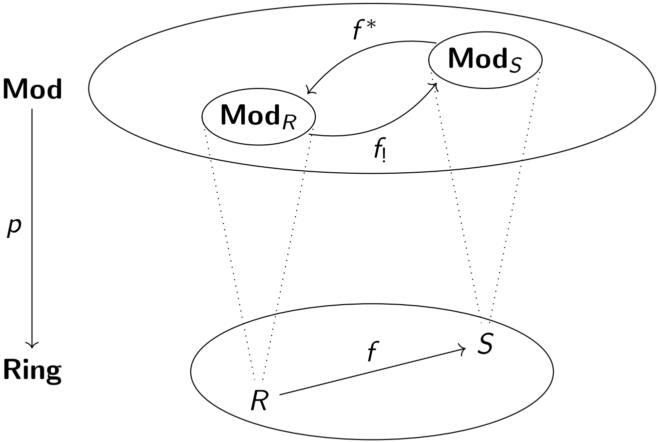

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p is also an *op*fibration: $f_!$ is extension of scalars $(- \otimes_R S)$

Example 3: the global *co*module category

$\mathbf{Comod}(\mathcal{V})$: pairs (M, C) where $M \in \mathbf{Comod}_C$, for C a comonoid.

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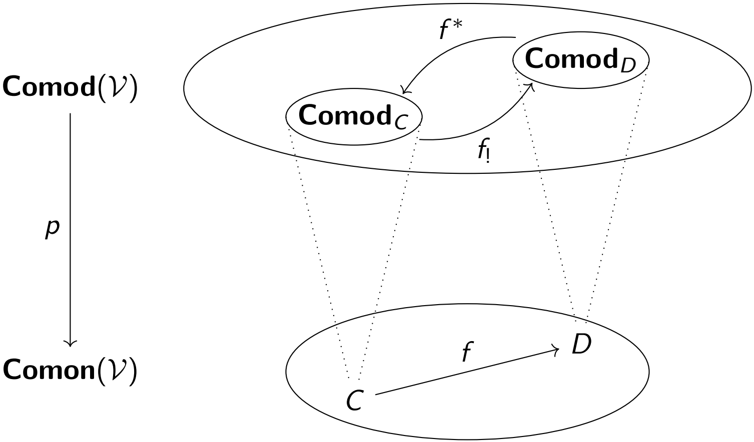

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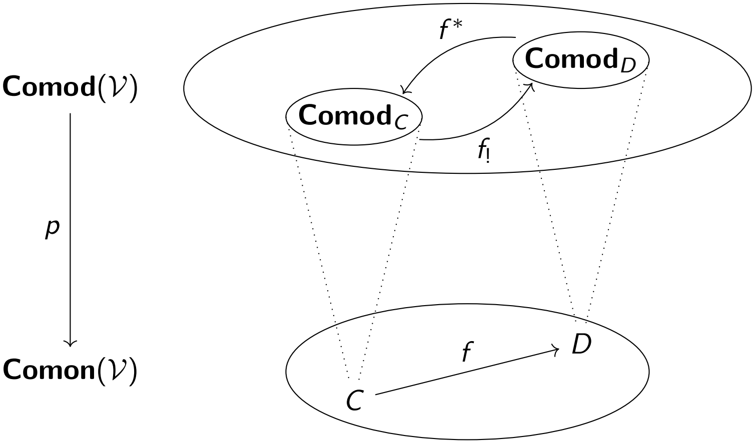

p is an opfibration: $f_!$ is corestriction of scalars

p is a fibration: f^* is coextension of scalars (given by cotensoring)

The Grothendieck construction

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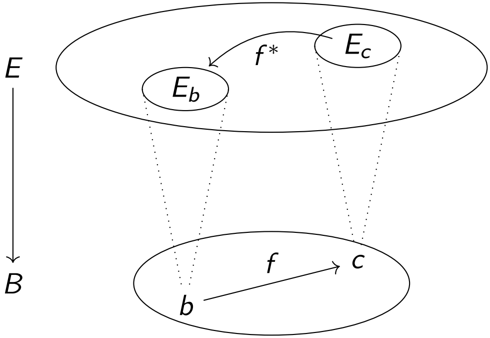

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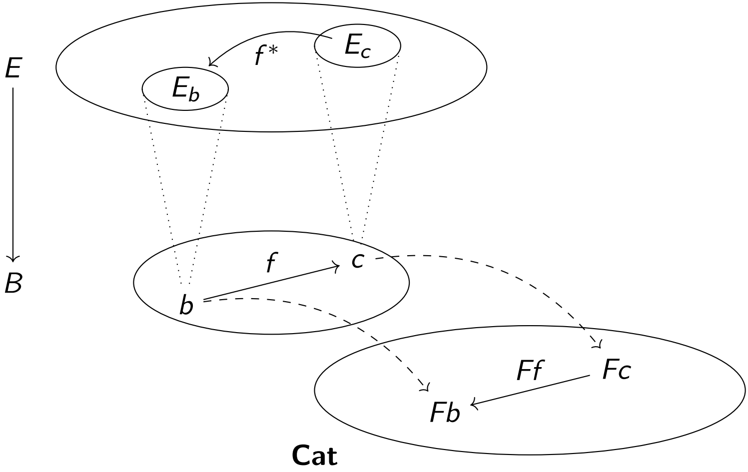

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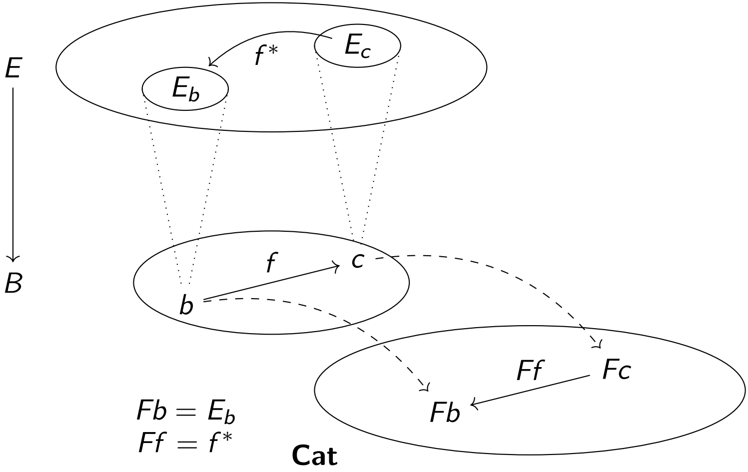

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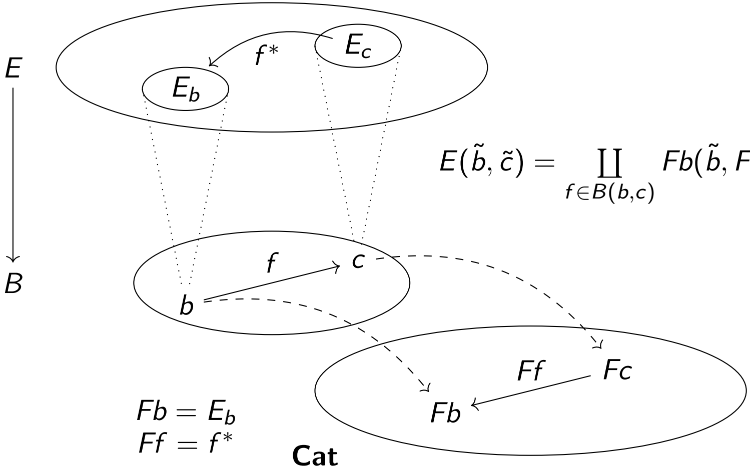

$$E(\tilde{b}, \tilde{c}) = \coprod_{f \in B(b,c)} Fb(\tilde{b}, Ff\,\tilde{c})$$

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Enriched categories

Capture the idea of a category whose homs $\mathcal{C}(x, y)$ have extra structure i.e. belong to a monoidal category $(\mathcal{V}, \otimes, \mathbf{1})$.

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$\times$	$\{*\}$	**Set**	categories
		Cat	strict 2-categories
		sSet	simplicial categories
$\otimes_k$	k	$\mathbf{Vect}_k$	k-linear categories
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Also, monoids in $\mathcal{V}$ are $\mathcal{V}$-categories with one object:

$$G \text{ a monoid}, \qquad \mathcal{C}(*, *) = G$$

Underlying categories and free $\mathcal{V}$-categories

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But every $\mathcal{V}$-category $\mathcal{C}$ has an **underlying category** $\mathcal{C}_0$:

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Conversely, there is often a **free $\mathcal{V}$-category** $C_{\mathcal{V}}$ on a category C, with the same objects and

$$C_{\mathcal{V}}(b, c) := \coprod_{f \in C(b,c)} \mathbf{1}$$

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One-object example when $\mathcal{V} = \mathbf{Vect}_k$:

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The maps $C \to (C_{\mathcal{V}})_0$ form the unit of the adjunction:

$$\begin{tikzcd} \mathbf{Cat} \arrow[r, bend left, "(-)_{\mathcal{V}}"] \arrow[r, phantom, "\perp"] & \mathcal{V}\text{-}\mathbf{Cat} \arrow[l, bend left, "(-)_0"] \end{tikzcd}$$

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Enriched fibrations

How can we define fibrations of enriched categories?

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Apply Street's definition of **fibration in a 2-category** to $\mathcal{V}$-**Cat**:

Definition

$p\colon \mathcal{E} \to \mathcal{B}$ is a **fibration** if $i\colon \mathcal{E} \hookrightarrow \mathcal{B}/p$ has a right adjoint over $\mathcal{B}$.

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, hook, bend left, "i"] \arrow[rr, phantom, "\perp"] \arrow[dr, "p"'] & & \mathcal{B}/p \arrow[ll, bend left, "r"] \arrow[dl, "\pi_{\mathcal{B}}"] \\
& \mathcal{B} &
\end{tikzcd}$$

Motivating the definition: Fibers

Fibers are given by pullback:

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Given $b \xrightarrow{f} c$, want a functor $\mathcal{E}_c \xrightarrow{f^*} \mathcal{E}_b$, but we can't get this from the universal property of pullbacks.

Motivating the definition: Comma categories

Instead of pullbacks, need comma objects:

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These requirements are satisfied by the defining adjunction:

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Applied to the 2-category **Cat**, we recover the classical definition.

Outline

The inverse Grothendieck construction

We have seen all the ingredients for:

Proposition (Beardsley-W.)

Let $\mathcal{B}$ be a $\mathcal{V}$-category. There is a 2-functor

$$\mathcal{V}\text{-}\mathbf{Fib}(\mathcal{B}) \to 2\text{-}\mathbf{Fun}({\mathcal{B}_0}^{op}, \mathcal{V}\text{-}\mathbf{Cat}).$$

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More generally, for a suitable 2-category $\mathcal{K}$, we should have

$$\mathbf{Fib}_{\mathcal{K}}(\mathcal{B}) \to 2\text{-}\mathbf{Fun}(\mathcal{B}_0^{op}, \mathcal{K}),$$

where $\mathcal{B}_0 := \mathcal{K}(\bullet, \mathcal{B})$.

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Outline

$\mathcal{V} = \mathbf{Vect}_k$

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Can't we say more about B-indexed $\mathcal{V}$-categories $B^{op} \to \mathcal{V}\text{-}\mathbf{Cat}$? Where does the Grothendieck construction land?

Another perspective

$$\mathbf{Set}\text{-}\mathbf{Fib}(B_{\mathbf{Set}}) \cong 2\text{-}\mathbf{Fun}(B^{op}, \mathbf{Set}\text{-}\mathbf{Cat}).$$

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Can we make this precise? What should **CoactFib** be?

Idea (Tamaki 2009)

We may not have maps $V \to \mathbf{1}$, but every V has a coaction by $\mathbf{1}$:

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When $\otimes = \times$, coactions correspond to maps $V \to C$, so coactions are ‘generalized maps’. $\leftarrow$ Can’t always be composed!

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What framework handles all these? The comodule bifibration!

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Cotensoring acts like pullback against a coaction, so this behaves like a category with pullbacks.

The comodule bifibration, categorified

Going up a dimension, we get:

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So we can define ‘fibrations’!

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Fibrations 'in' a 2-functor

For a 2-functor $P\colon \mathcal{M} \to \mathcal{K}$ admitting 'pullbacks' and 'commas', we may define the category P-**Fib** of P-**fibrations**.

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However, this is unlikely to be an equivalence:
Full comonoid structure is too much, we only need the counit.

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What other results are secretly about comonoids/comodules?

Thank you!

Questions/comments?