Cubes with connections from algebraic weak factorization systems

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(work in progress with Chris Kapulkin)

Western Topology Seminar
11 Nov 2020

Outline

- The interval, homotopies, and cubes
- Weak factorization systems
  - Functorial
  - Algebraic
- Cylinders from functorial and algebraic WFS

The Interval and Homotopies

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& X \times \mathbb{I} \arrow[r, "H"] & Y \\
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$\rightarrow$ first example of *weak equivalences* encountered in math

The Interval and Higher-dimensional Geometry

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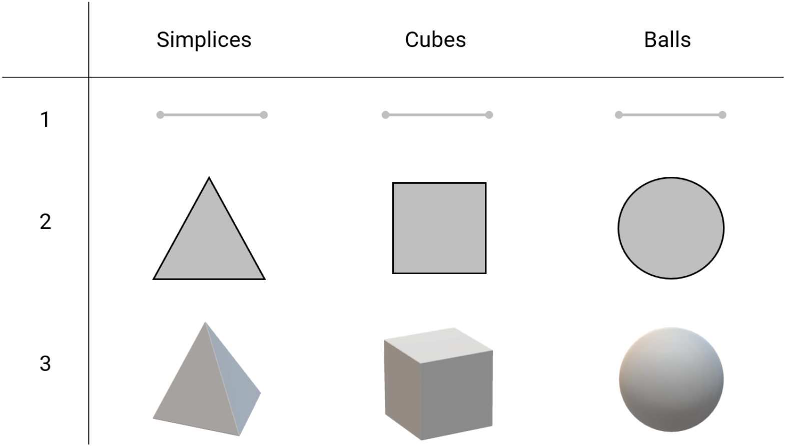

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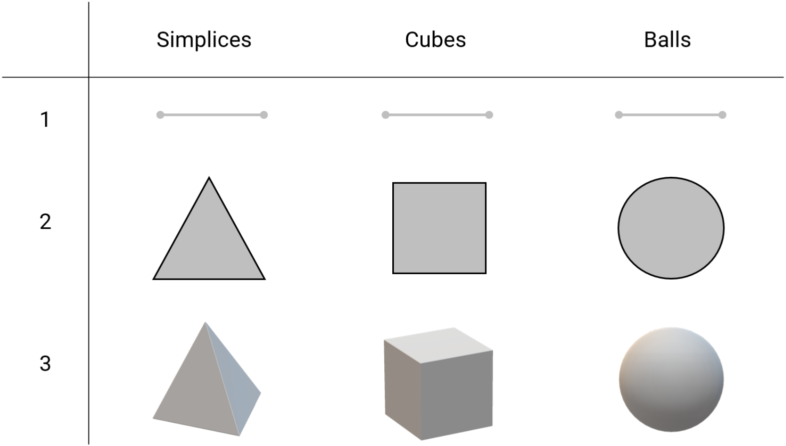

Today, we will focus on *cubes*.

Why cubes?

Proposition

Higher-dimensional cubes are more appealing and intuitive than simplices and balls.

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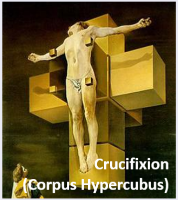
Proof.


A Wrinkle in Time


Crucifixion (Corpus Hypercubus)


Interstellar


Avengers: Infinity War

Simplices vs cubes

The *simplex category* Δ does have the advantage of simplicity:

- objects are $[n] = \{0 \leq 1 \leq \cdots \leq n\}$
- morphisms are all order-preserving maps
    - generated by faces and degeneracies

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By contrast, the *cube category* $\Box$ requires some choices:

- objects are $[1]^n = \{0 \leq 1\}^n$
- morphisms are some subset of order-preserving maps
  - generated by faces and degeneracies, and possibly ...
  - connections
  - symmetries
  - diagonals

Simplices vs cubes

Generators for maps in $\square$:

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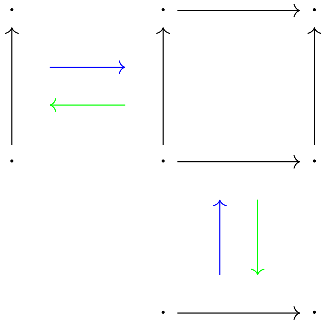
- *face* and *degeneracy* maps



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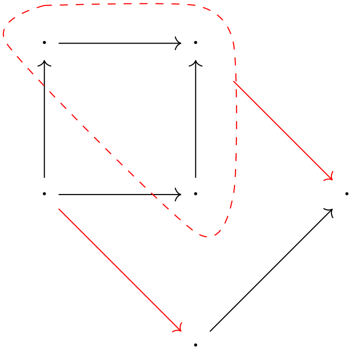

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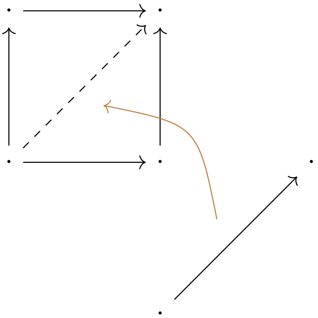

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Generators for maps in $\square$:

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$$\begin{tikzcd}
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\cdot \arrow[u] \arrow[r] & \cdot \arrow[u]
\end{tikzcd}$$

Cubes with connections

The case for faces, degeneracies *and connections*:

[1] Kapulkin-Lindsey-W. 2019

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- $\square$ is a generalized Reedy category

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- There is a co-reflective embedding[1] $\mathbf{sSet} \hookrightarrow \mathbf{cSet}$

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Cubes with connections

The case for faces, degeneracies *and connections*:

- $\square$ is a generalized Reedy category
- $\square$ is a strict test category
- There is a co-reflective embedding[1] $\mathbf{sSet} \hookrightarrow \mathbf{cSet}$
- It's exactly what we get from *algebraic* weak factorization systems (this talk)

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Interlude: Embedding **sSet** into **cSet**

As long as $\square$ has faces and degeneracies, we can define certain quotients of the standard cubes in **cSet**:

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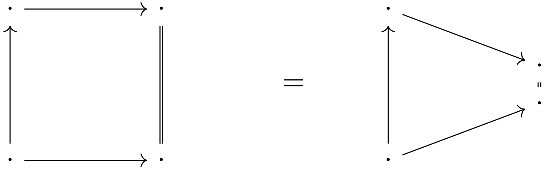
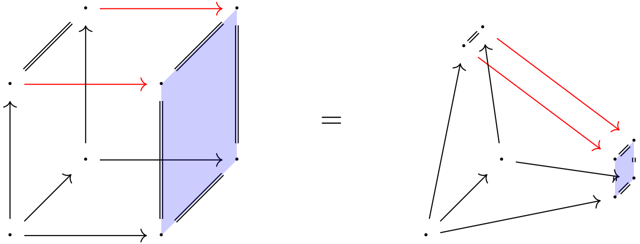
$$Q^0 \quad = \quad \cdot \qquad\qquad Q^1 \quad = \quad \cdot \longrightarrow \cdot$$

$Q^2 \quad =$



$Q^3 \quad =$



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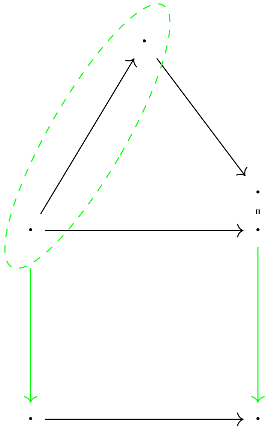

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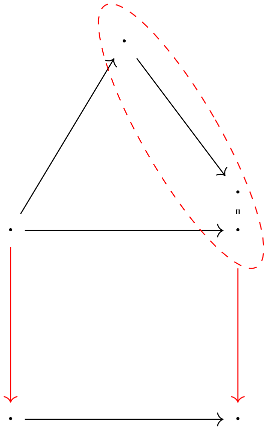

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(model categories)

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$$\begin{tikzcd}
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\text{classical homotopy theory} \arrow[d, squiggly] & & \text{cubes} \arrow[d, squiggly] \\
\text{abstract homotopy theory} \arrow[rr, dashed, dash, "?" description] & & \text{the cube category} \\
\text{(model categories)} & &
\end{tikzcd}$$

Model categories

A *model category* has all limits and colimits, and classes of:

- $\xrightarrow{\sim}$ *weak equivalences* satisfying 2-out-of-3
- $\rightarrowtail$ *cofibrations* (nice injections)
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such that we have *weak factorization systems*:

$$(\overset{\sim}{\rightarrowtail} \;,\; \twoheadrightarrow)$$

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Factorizations and lifts are *not unique*!

Weak Factorization Systems

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Intermediate versions:

$$\cdot \;\text{———}\; \cdot \;\text{———}\; \cdot \;\text{————}\; \cdot$$

weak functorial algebraic orthogonal

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weak	exist	exist
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algebraic		
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We will see how to get cylinders from these WFS

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$\mathrm{Cy}X$ is a substitute for $X \otimes \mathbb{I}$ in the absence of $\mathbb{I}$ or $\otimes$
e.g. can define homotopies using $\mathrm{Cy}X$

Cylinders from functorial WFS

If we have a *functorial* WFS, then Cy becomes a functor as well:

$$
\begin{tikzcd}
X \sqcup X \arrow[r, "f \sqcup f"] \arrow[d, "\nabla"'] & Y \sqcup Y \arrow[d, "\nabla"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
X \sqcup X \arrow[r, "f \sqcup f"] \arrow[d, tail] & Y \sqcup Y \arrow[d, tail] \\
\mathrm{Cy} X \arrow[r, "\mathrm{Cy} f"] \arrow[d, two heads, "\sim"] & \mathrm{Cy}\, Y \arrow[d, two heads, "\sim"'] \\
X \arrow[r, "f"] & Y
\end{tikzcd}
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And the left and right factorizations of ∇ are components of *face* and *degeneracy* natural transformations:

$$\partial_i \colon \mathrm{Id} \Rightarrow \mathrm{Cy} \qquad\qquad \sigma \colon \mathrm{Cy} \Rightarrow \mathrm{Id}$$

Cylinders from functorial WFS

Applying Cy to the components of ∂_i and σ, we get the *cubical identities* for faces and degeneracies:

$$
\begin{tikzcd}
X \sqcup X \arrow[r, "\partial_i \sqcup \partial_i"] \arrow[d, "\partial_0 \sqcup \partial_1"'] & \mathrm{Cy}X \sqcup \mathrm{Cy}X \arrow[r, "\sigma \sqcup \sigma"] \arrow[d, "\partial_0 \circ \mathrm{Cy} \sqcup \partial_1 \circ \mathrm{Cy}"] & X \sqcup X \arrow[d, "\partial_0 \sqcup \partial_1"] \\
\mathrm{Cy}X \arrow[r, "\mathrm{Cy} \circ \partial_i"] \arrow[d, "\sigma"'] & \mathrm{Cy}^2 X \arrow[r, "\mathrm{Cy} \circ \sigma"] \arrow[d, "\sigma \circ \mathrm{Cy}"] & \mathrm{Cy}X \arrow[d, "\sigma"] \\
X \arrow[r, "\partial_i"] & \mathrm{Cy}X \arrow[r, "\sigma"] & X
\end{tikzcd}
$$

$$\sigma\ \partial_i = \mathrm{Id}$$

$$(\partial_i \circ \mathrm{Cy})\ \partial_j = (\mathrm{Cy} \circ \partial_j)\ \partial_i$$

$$(\sigma \circ \mathrm{Cy})(\mathrm{Cy} \circ \partial_i) = \partial_i\ \sigma = (\mathrm{Cy} \circ \sigma)(\partial_i \circ \mathrm{Cy})$$

$$\sigma\ (\mathrm{Cy} \circ \sigma) = \sigma\ (\sigma \circ \mathrm{Cy})$$

Cylinders from functorial WFS

Lemma

A category $\mathcal{C}$ with coproducts and a functorial WFS has:

- *a cylinder functor* $\mathsf{Cy}\colon \mathcal{C} \to \mathcal{C}$
- *faces* $\partial_0, \partial_1 \colon \mathsf{Id} \Rightarrow \mathsf{Cy}$
- *degeneracies* $\sigma \colon \mathsf{Cy} \Rightarrow \mathsf{Id}$

satisfying the cubical identities for faces and degeneracies.

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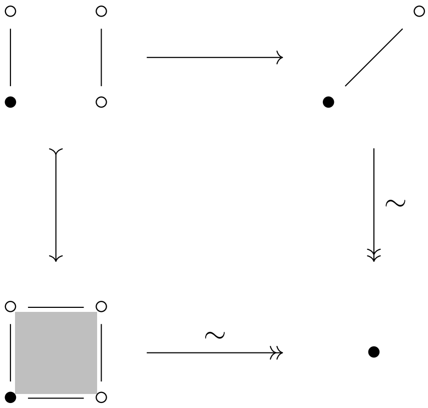
satisfying the cubical identities for faces and degeneracies.

Each $X \in \mathcal{C}$ gives a co-cubical object (without connections):

$$\begin{tikzcd} X \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] & \mathrm{Cy}X \arrow[r, shift left=5] \arrow[r, leftarrow, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, leftarrow, shift right=3] \arrow[r, shift right=5] & \mathrm{Cy}^2X \quad \cdots \quad \cdots \end{tikzcd}$$

Cylinders from functorial WFS

With the current setup (coproducts and functorial WFS), we can actually define connections:



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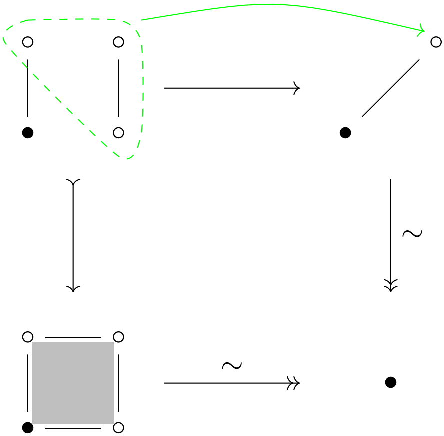

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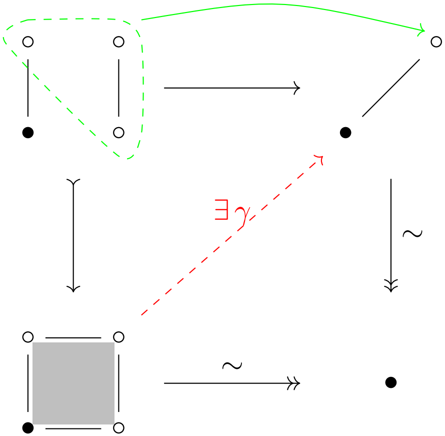

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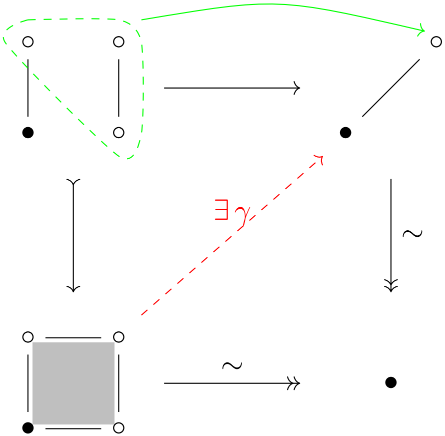

But we can't get some cubical identities without uniqueness of lifts.

Cylinders from algebraic WFS

In a functorial WFS, we have $L\colon \mathcal{C}^{[1]} \to \mathcal{C}^{[1]}$ and $R\colon \mathcal{C}^{[1]} \to \mathcal{C}^{[1]}$:

$$X \xrightarrow{f} Y \qquad \rightsquigarrow \qquad X \xrightarrow{Lf} \cdot \xrightarrow{Rf} Y$$

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Although we still don't have uniqueness of lifts in general, we have uniqueness of R-algebra homomorphisms out of *free R-algebras*.

Cylinders from algebraic WFS

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Although we still don't have uniqueness of lifts in general, we have uniqueness of R-algebra homomorphisms out of *free R-algebras*.

$\to$ this is enough to make connections satisfy cubical identities!

Cylinders from algebraic WFS

Proposition (Kapulkin-W.)

A category $\mathcal{C}$ with coproducts and an algebraic WFS has:

- *a cylinder functor* $\mathsf{Cy} \colon \mathcal{C} \to \mathcal{C}$
- *faces* $\partial_0, \partial_1 \colon \mathsf{Id} \Rightarrow \mathsf{Cy}$
- *degeneracies* $\sigma \colon \mathsf{Cy} \Rightarrow \mathsf{Id}$
- *connections* $\gamma_0, \gamma_1 \colon \mathsf{Cy}^2 \Rightarrow \mathsf{Cy}$

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Each $X \in \mathcal{C}$ gives a co-cubical object with connections:

$$X \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}\; \mathsf{Cy}X \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow}\; \mathsf{Cy}^2X \quad \cdots \quad \cdots$$

Cubical enrichment from algebraic WFS

Corollary

A category $\mathcal{C}$ with coproducts and an algebraic WFS is enriched over cubical sets, with

$$\mathcal{C}(X, Y)_n := \mathcal{C}(\mathrm{Cy}^n X, Y).$$

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Theorem (Garner '09)

Let $\mathcal{C}$ be a cocomplete category satisfying a 'smallness' condition (e.g. locally presentable). Then any cofibrantly generated WFS can be upgraded to an algebraic WFS.

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Corollary

Any category satisfying the above conditions is enriched over cubical sets.

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- If $\mathcal{C}$ is an *algebraic model category* (Riehl '11):
  - We get two enrichments: how are they related?
  - Is $\mathcal{C}$ an *enriched model category*?
- By quotienting cubes, can we make $\mathcal{C}$ *simplicially* enriched?

Thank you!

Background on functorial and algebraic WFS

Functorial factorizations

A *functorial factorization* on a category $\mathcal{C}$ is a functor

$$\mathcal{C}^{[1]} \longrightarrow \mathcal{C}^{[2]}$$

$$\begin{tikzcd} & x \arrow[d, "Lf"] \\ x \arrow[d, "f"'] & Ef \arrow[d, "Rf"] \\ y & y \end{tikzcd}$$

$$\begin{array}{c} x \\ {\scriptstyle f}\downarrow \\ y \end{array} \longmapsto \begin{array}{c} x \\ \downarrow{\scriptstyle Lf} \\ Ef \\ \downarrow{\scriptstyle Rf} \\ y \end{array}$$

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A WFS $(\mathcal{L}, \mathcal{R})$ is *functorial* if it has a functorial factorization with

$$Lf \in \mathcal{L} \qquad \text{and} \qquad Rf \in \mathcal{R}.$$

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Dually:

$L\colon \mathcal{C}^{[1]} \to \mathcal{C}^{[1]}$ is *co-pointed*, and we may ask if L is a *comonad*.

Algebraic weak factorization systems

Definition (Grandis-Tholen '06, Garner '07)

An *algebraic* WFS is a functorial WFS along with

- $\delta \colon L \Rightarrow LL$ making L a comonad
- $\mu \colon RR \Rightarrow R$ making R a monad
- such that $(\delta, \mu) \colon LR \Rightarrow RL$ is a distributive law

$$
\begin{tikzcd}
\cdot \arrow[r, equal] \arrow[d, "Lf"'] & \cdot \arrow[d, "LLf"] \\
\cdot \arrow[r, "\delta_f"] \arrow[d, "LRf"'] & \cdot \arrow[d, "RLf"] \\
\cdot \arrow[r, "\mu_f"'] \arrow[d, "RRf"'] & \cdot \arrow[d, "Rf"] \\
\cdot \arrow[r, equal] & \cdot
\end{tikzcd}
$$

R-algebras and L-coalgebras

Since R is a monad, we can talk about R-algebras: maps g equipped with an action $\alpha \colon Rg \to g$.

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$$
\begin{tikzcd}
\cdot \arrow[r, equal] \arrow[d, "f"'] & \cdot \arrow[r, equal] \arrow[d, "Lf"] & \cdot \arrow[d, "f"] \\
\cdot \arrow[r, "\kappa"] & \cdot \arrow[r, "Rf"] & \cdot
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\cdot \arrow[r, "Lg"] \arrow[d, "g"'] & \cdot \arrow[r, "\alpha"] \arrow[d, "Rg"] & \cdot \arrow[d, "g"] \\
\cdot \arrow[r, equal] & \cdot \arrow[r, equal] & \cdot
\end{tikzcd}
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\begin{tikzcd}
\cdot \arrow[r, equal] \arrow[d, "f"'] & \cdot \arrow[r, equal] \arrow[d, "Lf"] & \cdot \arrow[d, "f"] \\
\cdot \arrow[r, red, "\kappa"] & \cdot \arrow[r, "Rf"] & \cdot
\end{tikzcd}
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\cdot \arrow[r, equal] & \cdot \arrow[r, equal] & \cdot
\end{tikzcd}
$$

This gives *canonical* lifts of L-coalgebras against R-algebras!

$$
\begin{tikzcd}
\cdot \arrow[r] \arrow[d, "L\text{-coalg} \ni f"'] & \cdot \arrow[d, "g \in R\text{-alg}"] \\
\cdot \arrow[r] \arrow[ur, red, dashed] & \cdot
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\cdot \arrow[r] \arrow[d, "Lf"'] & \cdot \\
\cdot \arrow[r, red] \arrow[d, "Rf"'] \arrow[d, red, bend right, "\kappa"', phantom] & \cdot \arrow[d, "Rg"] \arrow[u, "Lg"'] \arrow[u, red, bend left, "\alpha", phantom] \\
\cdot \arrow[r] & \cdot
\end{tikzcd}
$$

Free algebras

We can do even better for *free* R-algebras: Rf for any f, with action given by $\mu_f \colon RRf \to Rf$.

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For any commuting square where g is an R-algebra:

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d, "f"'] & \cdot \arrow[d, "g \in R\text{-alg}"] \\
\cdot \arrow[r] & \cdot
\end{tikzcd}$$

there is a *unique* R-algebra map $\varphi \colon Rf \to g$:

$$\begin{tikzcd}
\cdot \arrow[rr] \arrow[dr, "Lf"] \arrow[dd, "f"'] & & \cdot \arrow[dd, "g"] \\
& \cdot \arrow[ur, dashed, red, "\varphi"] \arrow[dl, "Rf"] & \\
\cdot \arrow[rr] & & \cdot
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& \cdot \arrow[ur, dashed, red, "\varphi"] \arrow[dl, "Rf"] & \\
\cdot \arrow[rr] & & \cdot
\end{tikzcd}
$$

Warning: this is only unique *as an R-algebra map*.