The Grothendieck Construction and Relative Nerve

Liang Ze Wong
(w. Jonathan Beardsley)

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Western University
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Outline

- The Grothendieck construction
- ∞-categorical versions:
  - Unstraightening
  - The Relative Nerve
  - The **sSet**-enriched version
- Application: the Operadic Nerve of a monoidal category

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In fact, G and N don't have to be groups!
They can be monoids, categories, and even ∞-categories.

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Let C be a category. Given a functor $\varphi \colon C \to \mathbf{Cat}$

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$N_\bullet \rtimes_\varphi C$ is often denoted $\int \varphi$, and there is a functor $p : \int \varphi \to C$.

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$$\int : \mathbf{Cat}^C \longrightarrow \mathbf{Cat}_{/C}$$

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i.e. Fibrations are functors over C whose fibres vary functorially.

The Grothendieck Construction: Enriched Version

Going back to the construction: given $\varphi\colon C \to \mathbf{Cat}$

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$$\mathrm{Ob}\left(\int \varphi\right) = \coprod_{c \in C} \mathrm{Ob}(N_c)$$

$$\int \varphi((x,c),(y,d)) = \coprod_{c \xrightarrow{g} d} N_d(\varphi_g(x), y)$$

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This form generalizes easily to enriched categories, where $N_d(\varphi_g x, y)$ lives in a monoidal category $\mathcal{V}$ other than **Set**.

The Grothendieck Construction: Enriched Version

Theorem (Beardsley-W., 2018)

Let $(\mathcal{V}, \otimes, \mathbf{1})$ *be a monoidal category where:*

1. $\mathbf{1}$ *is terminal*
2. $\mathcal{V}$ *has pullbacks and coproducts*
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Non-examples: $\mathbf{Vect}_k, \mathbf{Ch}_k$

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∞-version: Unstraightening

Theorem (Lurie, 2009)

*For $S \in$ **sSet**, there is an equivalence of ∞-categories:*

$$\int_{\infty} : \mathbf{Cat_{\infty}}^{S} \xrightarrow{\simeq} \mathbf{coCart}_{/S}$$

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- Un_+ itself is not given explicitly, but as the adjoint to the *marked straightening* functor St_+
- But if $C \in \mathbf{Cat}$ and $f : C \to \mathbf{sSet}$ where each $f(c)$ is a quasicategory, we have an explicit for for $\int_\infty \mathrm{N}(f)$.

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- such that for all $I \subseteq J \subseteq [n]$ with max. elts. $i \leq j$,

$$\begin{array}{ccc} \Delta^I & \xrightarrow{s^I} & f(c_i) \\ \big\downarrow & & \big\downarrow {\scriptstyle f(c_{ij})} \\ \Delta^J & \xrightarrow{s^J} & f(c_j) \end{array}$$

Stricter ∞-versions

Theorem (Lurie, 2009)

Let $f\colon C \to \mathbf{sSet}$ *be a functor where each* $f(c)$ *is a quasicategory, so that* $\mathrm{N}(f)$ *is a map* $\mathrm{N}(C) \to \mathbf{Cat}_\infty$.

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$$\int_\infty \mathrm{N}(f) \simeq \mathrm{N}_f(C)$$

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We can strictify even further:

Theorem (Beardsley-W., 2018)

Let $F\colon C \to \mathbf{sCat}$ *be a functor where each* $F(c)$ *is locally Kan, and*

$$f\colon C \xrightarrow{F} \mathbf{sCat} \xrightarrow{\mathrm{N}} \mathbf{sSet}.$$

Then

$$\mathrm{N}_f(C) \simeq \mathrm{N}\left(\int_{\mathbf{sSet}} F\right).$$

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The Operadic Nerve

Definition (Lurie, 2007)

A **monoidal ∞-category** is a cocart. fibration $M \to \mathrm{N}(\Delta^{op})$ s.t.

$$M_{[n]} \simeq (M_{[1]})^n.$$

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Let C be a simplicial monoidal category. There is a simplicial category $C^{\otimes}$ along with a functor $C^{\otimes} \to \Delta^{op}$ such that

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Let C be a strict simplicial monoidal category. Then there is a functor $C^\bullet \colon \Delta^{op} \to \mathbf{sCat}$ sending $[n]$ to C^n, such that

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Corollary

Let f denote the composite $\Delta^{op} \xrightarrow{C^\bullet} \mathbf{sCat} \xrightarrow{\mathrm{N}} \mathbf{sSet}$. Then

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The result also holds for $\mathcal{O}$-monoidal categories: Replace $\mathcal{A}ssoc$ with a **Set**-operad $\mathcal{O}$ and Δ^{op} with the category of operators of $\mathcal{O}$.

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This gives a better handle on *co*algebras in $\mathrm{N}(C^{\otimes})$.

Thank you!

Questions/comments?