Cohen-Montgomery Duality and the Grothendieck Correspondence

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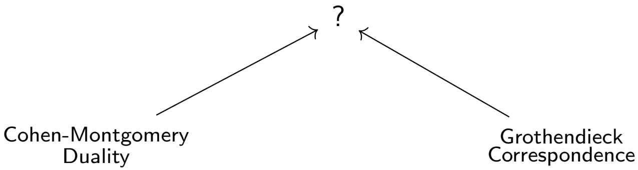
Plan

Cohen-Montgomery Duality

Grothendieck Correspondence

Plan



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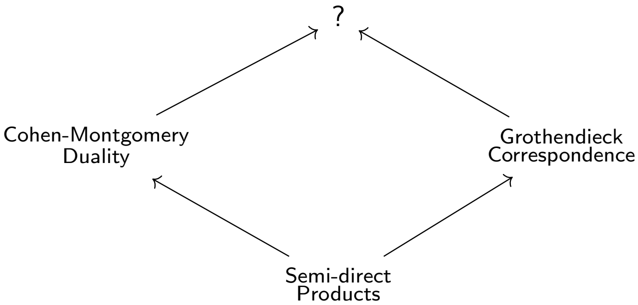
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Cohen-Montgomery Duality
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Semi-direct Products

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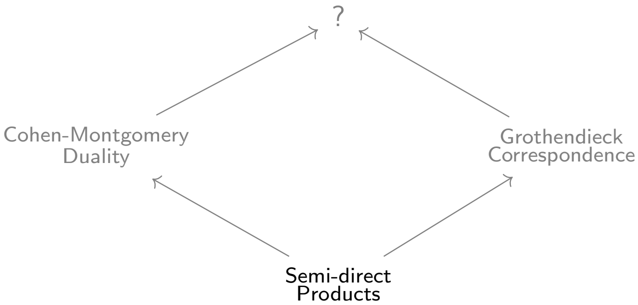
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Can form $N \rtimes_{\varphi} G$ = the set $N \times G$ with multiplication

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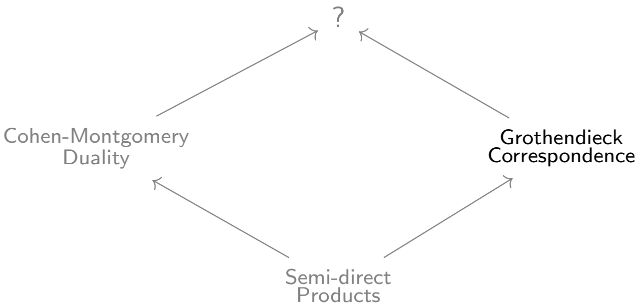
In fact, G and N don't have to be groups!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The Grothendieck Construction — $\rtimes C$

Given a small category C and a functor $\varphi\colon C^{op} \to \mathbf{Cat}$

$$c \mapsto N_c \in \mathbf{Cat} \qquad\qquad (c \xrightarrow{g} d) \mapsto (N_d \xrightarrow{\varphi_g} N_c)$$

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This has a functor $N_\bullet \rtimes_\varphi C \to C$.

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and $N_\bullet \rtimes_\varphi C$ is a category with one object $\cdot$ and

$$\mathrm{Hom}(\cdot,\cdot) = N \rtimes_\varphi G.$$

The Grothendieck Correspondence

Theorem (Grothendieck, 1959)

Let C be a category. There is an adjunction

$$\mathbf{Cat}_{/C} \underset{- \rtimes C}{\overset{- \swarrow C}{\rightleftarrows}} \mathbf{Fun}(C^{op}, \mathbf{Cat})$$

$$\bot$$

which is an equivalence if C is a groupoid.

The Comma Category $- \swarrow C$

Given a functor $p\colon X \to C$ and $c \in C$, define the category $X \swarrow c$:

- objects are pairs (x, g) where $x \in X$ and $g\colon c \to px$

$$\begin{array}{ccc} & & x \\ & & \big\downarrow{\scriptstyle p} \\ c & \xrightarrow{\;g\;} & px \end{array}$$

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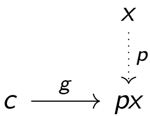
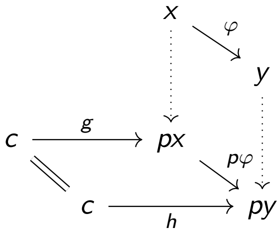


- maps $(x, g) \to (y, h)$ are maps $\varphi\colon x \to y$ s.t. $(p\varphi)g = h$



The Comma Category $- \swarrow C$

When $X = \cdot \circlearrowleft^{E}$ and $C = \cdot \circlearrowleft^{G}$, we get $X \swarrow \cdot$ where:

- objects are $g \in G$

$$\begin{array}{ccc} & & x \\ & & \vdots \\ & & \downarrow \\ c & \xrightarrow{g} & px \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} & & \cdot \\ & & \vdots \\ & & \downarrow \\ \cdot & \xrightarrow{g} & \cdot \end{array}$$

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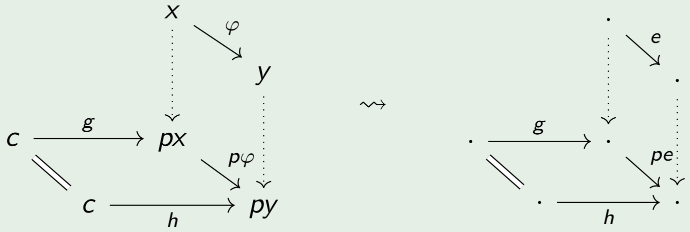

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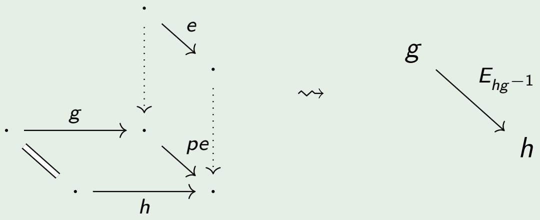

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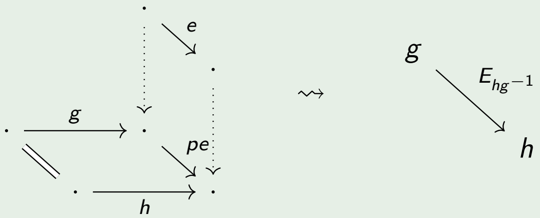

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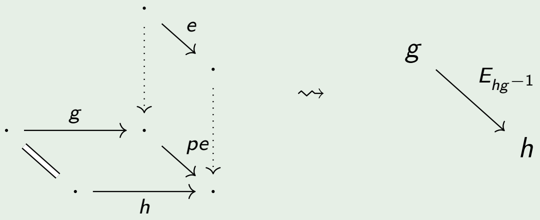

Note: If $E = N \rtimes G$, then $E_{hg^{-1}} \cong N$.

Combining $- \swarrow C$ and $- \rtimes C$

Theorem (Grothendieck, 1959)

Let C be a category. There is an adjunction

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which is an equivalence if C is a groupoid.

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Let $C = \cdot \circlearrowleft G$ and $\varphi \colon C^{op} \to \mathbf{Cat}$ be induced by a G-action.

$\cdot \circlearrowleft N$

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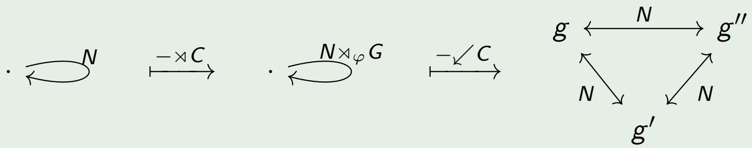

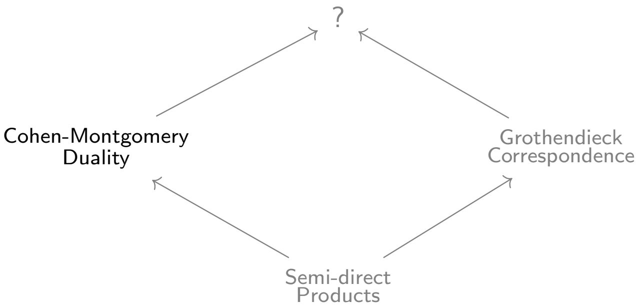
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Maps of sets $p\colon X \to G$ are the same as G-gradings on X:

$$X = \coprod_{g \in G} X_g \qquad\qquad X_g := p^{-1}(g)$$

Cohen-Montgomery Duality

Theorem (Cohen & Montgomery, 1984)

Let G be a finite group, $|G| = n$. There are functors

$$\mathbf{Alg}_G \underset{-\rtimes G}{\overset{-\#kG^*}{\rightleftarrows}} G\text{-}\mathbf{Alg}$$

such that $(A \rtimes G)\#kG^* \cong M_n(A)$ *and* $(A\#kG^*) \rtimes G \cong M_n(A)$.

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If $E = A \rtimes G$, then $E_{hg^{-1}} = A$, so $E\#kG^* = M_n(A)$.

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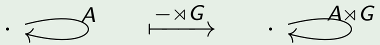

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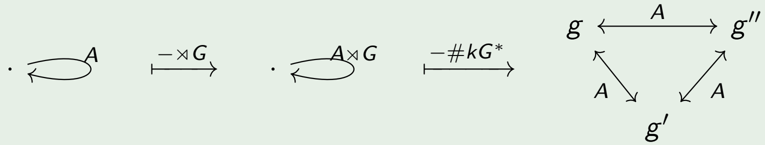

Generalizing Cohen-Montgomery Duality I

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1984 Cohen-Montgomery	$H = kG$, G finite
1984 van den Bergh	H Hopf algebra, finite
1985 Blattner-Montgomery	H Hopf algebra
1999 Nikshych	H weak Hopf algebra, finite

Generalizing Cohen-Montgomery Duality II

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Generalizing Cohen-Montgomery Duality II

G-graded algebras	$\leftrightarrow$	algebras with G-action
G-graded categories	$\leftrightarrow$	categories with G-action

1984 Cohen-Montgomery	G finite group, A has 1 object
2006 Cibils-Marcos	G group, A has ∞ objects
2008 Lowen	G category, A has ∞ objects

Cohen-Montgomery + Grothendieck = ?

G-graded algebras	$\leftrightarrow$	algebras with G-action
H-comodule categories	$\leftrightarrow$	H-module categories

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	H	Hopf	dim	ObH	ObA	k-linear
$\rtimes$	G	✓	∞	1	1	
'59 Groth	G	(✓)	∞	∞	∞	
'84 Coh-Mon	kG	✓	finite	1	1	✓

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'84 vdBer	H	✓	finite	1	1	✓
'85 Bla-Mon	H	✓	∞	1	1	✓
'99 Niksh	H	✓	finite	n	1	✓
'06 Cib-Mar	G	✓	∞	1	∞	✓
'08 Lowen	G	(✓)	∞	∞	∞	✓

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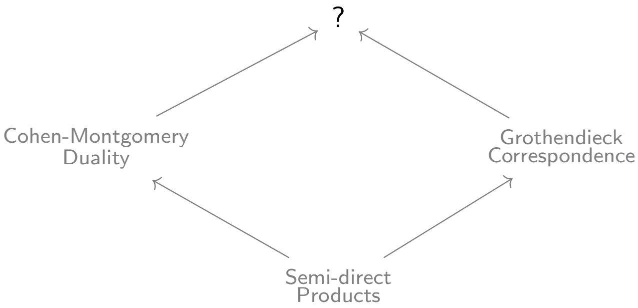
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Want to combine the two generalizations of semi-direct products:

- from groups to categories (one object to many objects)
- from groups to algebras (linearizing)

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More generally, we can work in a monoidal category $(\mathcal{V}, \otimes, \mathbf{1})$ and ask for 'many-object algebras' in $\mathcal{V}$.

Cohen-Montgomery + Grothendieck = ?

The ‘category’ A being (co)acted on should be:

Definition (Aguiar, 1997)

A $\mathcal{V}$-**internal category** is $A = (A_0, A_1)$ where:

- A_0 is a coalgebra in $\mathcal{V}$, with a bi-coaction on A_1
- A_1 is an algebra in (A_0, A_0)-**Bicomod**

$$A_0 \xrightarrow{u} A_1 \qquad\qquad A_1 \boxtimes_{A_0} A_1 \xrightarrow{m} A_1$$

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$$A_0 \xrightarrow{u} A_1 \qquad\qquad A_1 \boxtimes_{A_0} A_1 \xrightarrow{m} A_1$$

Any k-algebra A gives rise to a $\mathbf{Vect}_k$-internal category (k, A).

Cohen-Montgomery + Grothendieck = ?

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Definition (Day & Street, 2003)

A $\mathcal{V}$-**quantum category** is a category $H = (H_0, H_1)$ where:

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Theorem (W., *in progress*)

Let $\mathcal{V}$ be a monoidal category with equalizers preserved by $\otimes$. For $H = (H_0, H_1)$ a $\mathcal{V}$-quantum category, there is an adjunction

$$H\text{-}\mathbf{ComodCat} \underset{- \rtimes H}{\overset{- \swarrow H}{\rightleftarrows}} H\text{-}\mathbf{ModCat} \quad (\perp)$$

which becomes an equivalence if H is a quantum groupoid.

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- Cohen-Montgomery Duality: $\mathcal{V} = \mathbf{Vect}_k$ and $H_0, A_0 = k$
- Grothendieck Correspondence: $\mathcal{V} = \mathbf{Set}$

Thank you!

Questions/comments?