Weak equivalences between categories of models of type theory

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The **Internal Language Conjectures** make precise the belief that intensional type theory is the internal language of ∞-categories:

Conjecture (Kapulkin-Lumsdaine 2016)

There are ∞-equivalences:

$$\begin{tikzcd}
\mathbf{Ctx}_{\mathsf{HoTT}} \arrow[r, dashed, "\sim"] \arrow[d, dashed] & \mathbf{ElemTopos}_\infty \arrow[d, dashed] \\
\mathbf{Ctx}_{\mathsf{Id},\Sigma,\Pi_{\mathsf{ext}}} \arrow[r, "\sim"] \arrow[d] & \mathbf{LCCC}_\infty \arrow[d] \\
\mathbf{Ctx}_{\mathsf{Id},\Sigma} \arrow[r, "\sim"] & \mathbf{Lex}_\infty
\end{tikzcd}$$

Chris Kapulkin and Peter LeFanu Lumsdaine. 'The homotopy theory of type theories'. In: *arXiv preprint arXiv:1610.00037* (2016).

Introduction

In this talk, we present progress towards the ∞-equivalence:

$$\mathbf{Ctx}_{\mathsf{Id},\Sigma} \longrightarrow \mathbf{Lex}_\infty$$

Chris Kapulkin and Karol Szumilo. 'Internal language of finitely complete $(\infty, 1)$-categories'. In: *arXiv preprint arXiv:1709.09519* (2017).

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$$
\begin{tikzcd}
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\mathbf{Ctx}_{\mathsf{Id},\Sigma} \arrow[r] & \mathbf{Trb} \arrow[r, "\sim", "\text{[KS17]}"'] & \mathbf{Fib} \arrow[r, "\sim", "\text{[S14]}"'] & \mathbf{Lex}_\infty
\end{tikzcd}
$$

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\mathbf{Ctx}_{\mathsf{Id},\Sigma} \arrow[r] & \mathbf{CwA}_{\mathsf{Id},\Sigma} \arrow[r] & \mathbf{Comp}_{\mathsf{Id},\Sigma} \arrow[r, "\sim", "\text{[KS17]}"'] & \mathbf{Trb} &
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Introduction

Theorem (CKNW)

Assume our models satisfy the Logical Framework (LF) *of* [LW15]. *The comparison functors*

$$\mathbf{Ctx}_{\mathsf{Id},\Sigma,\mathsf{LF}} \hookrightarrow \mathbf{CwA}_{\mathsf{Id},\Sigma,\mathsf{LF}} \hookrightarrow \mathbf{Comp}_{\mathsf{Id},\Sigma,\mathsf{LF}} \hookleftarrow \mathbf{Trb}_{\mathsf{LF}}$$

have homotopy inverses, hence are ∞*-equivalences.*

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Plan:

- Compare these models of type theory
- Highlight the role that each model plays in the equivalence
- Compare weak equivalences within each model

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Type-theoretic models

$$\mathbf{Ctx} \hookrightarrow \mathbf{CwA} \hookrightarrow \mathbf{Comp} \hookleftarrow \mathbf{Trb}$$

Modelling type theories with categories

a type theory	$\mathbb{T}$	$\mathcal{C}$	a category
contexts	$\Gamma, \Delta, \ldots$	$\Gamma, \Delta, \ldots$	objects
substitutions	$\Delta \xrightarrow{\sigma} \Gamma$	$\Delta \xrightarrow{\sigma} \Gamma$	maps

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substitution of A along $\Delta \xrightarrow{\sigma} \Gamma$	$A[\sigma]$	$\Delta.A[\sigma]$	pullback of $\Gamma.A \twoheadrightarrow \Gamma$ along $\Delta \xrightarrow{\sigma} \Gamma$

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Further, choice of pullbacks is functorial.

Encoding type dependencies

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Contextual categories have an $\mathbb{N}$-grading on objects

$$\mathrm{Ob}_0\mathcal{C} \longleftarrow \cdots \longleftarrow \mathrm{Ob}_n\mathcal{C} \xleftarrow{p_n} \mathrm{Ob}_{n+1}\mathcal{C} \longleftarrow \cdots$$

such that $p_n(\Gamma.A) = \Gamma$ (where $\Gamma \in \mathrm{Ob}_n\mathcal{C}$).

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Categories with Attributes have a functor

$$\mathrm{Ty}\colon \mathcal{C}^{op} \to \mathbf{Set}$$

such that $A \in \mathrm{Ty}(\Gamma)$.

Homotopy-theoretic models

$$\mathbf{Ctx} \hookrightarrow \mathbf{CwA} \hookrightarrow \mathbf{Comp} \hookleftarrow \mathbf{Trb}$$

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Tribes are categories with weak equivs. $:=$ homotopy equivs.

Contextual categories vs. Tribes

Contextual categories are tribes:

	Contextual categories	**Tribes**
$\twoheadrightarrow$	Canonical projections	Fibrations
Σ	Dependent sum	Composition
Id	Identity types	Path objects

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However, tribes are not contextual categories:

- no functorial choice of pullbacks
- no pullback-stable choice of Id- and Σ-types.

$$\mathbf{Ctx} \hookrightarrow \mathbf{CwA} \hookrightarrow \mathbf{Comp} \hookleftarrow \mathbf{Trb}$$

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Theorem (Lumsdaine-Warren 2015)

Let $\mathcal{C}$ be a full comprehension category satisfying LF, *with weakly stable* Id *and* Σ*-types.*

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Logical Framework (LF):

- Existence of categorical dependent exponentials.
- Satisfied if $\mathcal{C}$ has Π-types or is locally cartesian closed.

$$
\begin{tikzcd}
\mathbf{Ctx} \arrow[r, hook] \arrow[r, phantom, shift right=2, "\perp"] & \mathbf{CwA} \arrow[l, bend left] \arrow[r, hook] \arrow[r, phantom, shift right=2, "\perp"] & \mathbf{SplComp} \arrow[l, bend left] \arrow[r, hook] \arrow[r, phantom, shift left=2, "\perp"] & \mathbf{Comp} \arrow[l, bend right, "{[LW15]}"'] \arrow[r, bend left] \arrow[r, phantom, shift left=2, "\perp"] & \mathbf{Trb} \arrow[l, hook']
\end{tikzcd}
$$

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\mathbf{Ctx} \arrow[r, hook] & \mathbf{CwA} \arrow[l, bend left, "\perp"'] \arrow[r, hook] & \mathbf{SplComp} \arrow[l, bend left, "\perp"'] \arrow[r, hook] & \mathbf{Comp} \arrow[l, bend right, "{[\mathrm{LW15}]}"', "\perp"] \arrow[r, bend left, "\perp"'] & \mathbf{Trb} \arrow[l, hook']
\end{tikzcd}$$

Need units and counits to be natural weak equivalences.

Weak equivalences within each model

Type-theoretic vs. homotopy-theoretic equivalences

Given Id-types/path objects, we may define homotopies $f \sim g$ and homotopy equivalences $X \xrightarrow{\sim} Y$.

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$F\colon \mathcal{C} \to \mathcal{D}$ is a **logical equivalence** (LE) if *types* and *terms* in $\mathcal{D}$ have preimages in $\mathcal{C}$, up to homotopy (*weak type* and *term lifting*).

$F\colon \mathcal{C} \to \mathcal{D}$ is a **homotopy-theoretic equivalence** (HE) if it induces an equivalence $\mathrm{Ho}\,\mathcal{C} \cong \mathrm{Ho}\,\mathcal{D}$.

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Note: LE assumes homotopies factor through chosen Id-types, and requires knowledge of 'immediate' type-dependencies.

Generalizations of logical equivalence

We may generalize LE in two ways, with a view towards HE:

- Require homotopies/homotopy equivalences to factor through *some* Id-type (rather than a chosen Id-type)
- Require weak *context* and *section* lifting

Generalizations of logical equivalence

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- Require homotopies/homotopy equivalences to factor through *some* Id-type (rather than a chosen Id-type)
- Require weak *context* and *section* lifting

	Type/term lifting	Context/section lifting
Chosen Id-type	LE	CE
Some Id-type	wLE	wCE

Summary

$$
\begin{tikzcd}[column sep=large]
(\text{Assume Id},\Sigma,{\color{red}\text{LF}}) & \text{LE} & \text{wLE} & \text{wCE} & \text{HE} \\
\mathbf{Ctx} \arrow[d, hook, shift right=2] \arrow[d, phantom, "\dashv"] & \bullet \arrow[rrr, equal, "{[KL16]}"'] \arrow[d, "\sim"'] & & & \bullet \arrow[dddd, dashed, "\sim"] \\
\mathbf{CwA} \arrow[u, bend right, shift right=2] \arrow[d, hook, shift right=2] \arrow[d, phantom, "\dashv"] & \bullet \arrow[d, "\sim"'] & & & \\
\mathbf{SplComp} \arrow[d, hook, shift left=2] \arrow[d, phantom, "\dashv"] & \bullet \arrow[r, equal] & \bullet \arrow[d, "\sim"'] & & \\
\mathbf{Comp} \arrow[u, bend left, shift right=2, "{[LW15]}"] \arrow[d, bend right, shift right=2] \arrow[d, phantom, "\dashv"] & & \bullet \arrow[r, equal] & \bullet \arrow[d, "\sim"', "{[KS17]}"] & \\
\mathbf{Trb} \arrow[u, hook, shift left=2] & & & \bullet \arrow[r, equal] & \bullet
\end{tikzcd}
$$

Thank you!

Questions/comments?

Categorical Semantics

- **Contexts** are finite lists $[x_1 : A_1, \ldots, x_n : A_n]$, up to definitional equality and renaming of variables
- **Substitutions** $[x_1 : A_1, \ldots, x_n : A_n] \xrightarrow{f} [y_1 : B_1, \ldots, y_m : B_m]$ are sequences

$$x_1 : A_1, \ldots, x_n : A_n \vdash f_1 : B_1$$

$$\vdots$$

$$x_1 : A_1, \ldots, x_n : A_n \vdash f_m : B_m$$

Contextual categories

Definition (Cartmell 1978)

A **contextual category** is a category $\mathcal{C}$ with:

1. A grading on objects $\mathrm{Ob}\,\mathcal{C} = \coprod_{n\in\mathbb{N}} \mathrm{Ob}_n\mathcal{C}$
2. A terminal object 1 which is the unique object in $\mathrm{Ob}_0\mathcal{C}$
3. Maps $\mathrm{pt}\colon \mathrm{Ob}_{n+1}\mathcal{C} \to \mathrm{Ob}_n\mathcal{C}$
4. *Canonical projections* $A \twoheadrightarrow \mathrm{pt}A$
5. A functorial choice of pullbacks against canonical projections:

$$\begin{tikzcd} A[\sigma] \arrow[r, "\sigma.A"] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, two heads] \\ \Delta \arrow[r, "\sigma"] & \mathrm{pt}A \end{tikzcd}$$

JW Cartmell. 'Generalised algebraic theories and contextual categories'. PhD thesis. University of Oxford, 1978.

Categories with Attributes

Definition (Cartmell 1978)

A **category with attributes** is a category $\mathcal{C}$ with:

1. A terminal object 1
2. A functor $\mathrm{Ty}\colon \mathcal{C}^{op} \to \mathbf{Set}$
3. For each $A \in \mathrm{Ty}(\Gamma)$, an object $\Gamma.A \in \mathcal{C}$ and map $\Gamma.A \twoheadrightarrow \Gamma$
4. A functorial choice of a map $\sigma.A$, for each $A \in \mathrm{Ty}(\Gamma)$ and $\Delta \xrightarrow{\sigma} \Gamma$, fitting into a pullback square:

$$\begin{tikzcd} \Delta.(\mathrm{Ty}f)(A) \arrow[r, "\sigma.A"] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma.A \arrow[d, two heads] \\ \Delta \arrow[r, "\sigma"] & \Gamma \end{tikzcd}$$

Tribes

Definition (Joyal 2014, Shulman 2015)

A **tribe** is a category $\mathcal{C}$ with a subcategory $\mathcal{F}$ of *fibrations* ($\twoheadrightarrow$) containing all isomorphisms, such that

1. $\mathcal{C}$ has a terminal object, and all objects are fibrant
2. Pullbacks along fibrations exist, and fibrations are stable under pullback
3. Anodyne maps ($\overset{\sim}{\rightarrowtail}$) are stable under pullback along fibrations
4. Every map factors as $\cdot \overset{\sim}{\rightarrowtail} \cdot \twoheadrightarrow \cdot$

Anodyne maps are those with the left lifting property against fibrations.

André Joyal. 'Categorical homotopy type theory'. In: *Slides from a talk at MIT dated* 17 (2014).

Michael Shulman. 'Univalence for inverse diagrams and homotopy canonicity'. In: *Mathematical Structures in Computer Science* 25.5 (2015), pp. 1203–1277.

Paths and homotopies within tribes

Definition

A **path object** for $Y \in \mathcal{C}$ is a factorization of the diagonal:

$$\Delta_Y = (Y \overset{\sim}{\rightarrowtail} PY \twoheadrightarrow Y \times Y)$$

Two maps $f, g \colon X \to Y$ are **homotopic** ($f \sim g$) if there is a factorization of (f, g) through PY:

$$(f, g) = (X \xrightarrow{H} PY \twoheadrightarrow Y \times Y)$$

A map $f \colon X \to Y$ is a **homotopy equivalence** ($f \colon X \xrightarrow{\sim} Y$) if there is a map $g \colon Y \to X$ such that $gf \sim 1_X$ and $fg \sim 1_Y$.

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Every tribe is a category with weak equivalences, where

$$\text{Weak equivalences} := \text{Homotopy equivalences}$$

Comprehension categories

Definition (Jacobs 1993)

A **comprehension category** consists of a Grothendieck fibration $\mathcal{T} \xrightarrow{P} \mathcal{C}$ and a *comprehension functor* $\chi \colon \mathcal{T} \to \operatorname{Arr}\mathcal{C}$ such that

$$\begin{tikzcd} \mathcal{T} \arrow[rr, "\chi"] \arrow[dr, "P"'] & & \operatorname{Arr}\mathcal{C} \arrow[dl, "\mathsf{cod}"] \\ & \mathcal{C} & \end{tikzcd}$$

commutes, and χ sends cartesian maps to pullback squares.
A comprehension category is **split** if P is a split fibration, and **full** if χ is fully faithful.

Categories with attributes are full split comprehension categories.

Bart Jacobs. 'Comprehension categories and the semantics of type dependency'. In: *Theoretical Computer Science* 107.2 (1993), pp. 169–207.

Homotopy equivalence

Let $\mathcal{C}$ be a model of type theory, with Id-types which give rise to homotopies $f \sim g$.

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Definition

The **homotopy category** of $\mathcal{C}$ is the category $\mathrm{Ho}\,\mathcal{C}$ with the same objects, and homotopy classes of maps.

Definition

A functor of models $F\colon \mathcal{C} \to \mathcal{D}$ is a **homotopy equivalence** (HE) if it induces an equivalences of categories $\mathrm{Ho}\,\mathcal{C} \cong \mathrm{Ho}\,\mathcal{D}$.

Logical equivalence

Definition

A functor $F\colon \mathcal{C} \to \mathcal{D}$ is a **logical equivalence** (LE) if it satisfies:

1. **weak type lifting**: For all $\Gamma \in \mathcal{C}, A \in \mathrm{Ty}(F\Gamma)$, there exists $\tilde{A} \in \mathrm{Ty}(\Gamma)$ and $F\tilde{A} \xrightarrow{\sim} A$ over $F\Gamma$.

$$\begin{tikzcd} & A \arrow[d, two heads] \\ \Gamma \arrow[r, dotted] & F\Gamma \end{tikzcd}$$

2. **weak term lifting**: For all $\Gamma \in \mathcal{C}, A \in \mathrm{Ty}(\Gamma)$ and $a \in \mathrm{Tm}(FA)$, there exists a term $\tilde{a} \in \mathrm{Tm}(A)$ such that $F\tilde{a} \sim a$.

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$$\begin{tikzcd} \tilde{A} \arrow[r, dotted, mapsto] \arrow[d, two heads] & F\tilde{A} \arrow[r, "\sim"] \arrow[dr, two heads] & A \arrow[d, two heads] \\ \Gamma \arrow[rr, dotted, mapsto] & & F\Gamma \end{tikzcd}$$

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2. **weak term lifting**: For all $\Gamma \in \mathcal{C}, A \in \mathrm{Ty}(\Gamma)$ and $a \in \mathrm{Tm}(FA)$, there exists a term $\tilde{a} \in \mathrm{Tm}(A)$ such that $F\tilde{a} \sim a$.

$$\begin{tikzcd} A \arrow[r, dotted] & FA \\ \Gamma \arrow[r, dotted] & F\Gamma \arrow[u, bend right, "a"'] \end{tikzcd}$$

Logical equivalence

Definition

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