Weak equivalences between categories of models of type theory

Simon Cho  Cory M. Knapp  Clive Newstead
Liang Ze Wong

Mentors:  Chris Kapulkin  Emily Riehl

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The **Internal Language Conjectures** make precise the belief that intensional type theory is the internal language of $\infty$-categories:

**Conjecture (Kapulkin-Lumsdaine 2016)**

There are $\infty$-equivalences:
In this talk, we present progress towards the $\infty$-equivalence:

$$\text{Ctx}_{\text{Id}, \Sigma} \longrightarrow \text{Lex}_\infty$$

---


In this talk, we present progress towards the $\infty$-equivalence:

\[
\begin{align*}
\text{Ctx}_{\text{Id}, \Sigma} \xrightarrow{\sim} \text{Trb} \xrightarrow{\sim} \text{Fib} \xrightarrow{\sim} \text{Lex}_\infty
\end{align*}
\]


In this talk, we present progress towards the $\infty$-equivalence:

$$\text{Ctx}_{\text{Id}, \Sigma} \xrightarrow{\sim} \text{Trb} \quad \xrightarrow{\sim} \quad \text{Fib} \quad \xrightarrow{\sim} \quad \text{Lex}_\infty$$

$$\text{Ctx}_{\text{Id}, \Sigma} \quad \xrightarrow{\sim} \quad \text{CwA}_{\text{Id}, \Sigma} \quad \xrightarrow{\sim} \quad \text{Comp}_{\text{Id}, \Sigma} \quad \xrightarrow{[\text{KS}17]} \quad \text{Trb}$$

---


Theorem (CKNW)

Assume our models satisfy the Logical Framework (LF) of [LW15].

The comparison functors

\[
\begin{align*}
\text{Ctx}_{\text{Id},\Sigma,\text{LF}} & \overset{\sim}{\longrightarrow} \text{CwA}_{\text{Id},\Sigma,\text{LF}} & \overset{\sim}{\longrightarrow} \text{Comp}_{\text{Id},\Sigma,\text{LF}} & \overset{\sim}{\longrightarrow} \text{Trb}_{\text{LF}}
\end{align*}
\]

have homotopy inverses, hence are $\infty$-equivalences.
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Assume our models satisfy the Logical Framework (LF) of [LW15].
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\]

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Plan:

- Compare these models of type theory
- Highlight the role that each model plays in the equivalence
- Compare weak equivalences within each model

---

Type-theoretic models

Ctx ← CwA → Comp ← Trb
Modelling type theories with categories

<table>
<thead>
<tr>
<th>a type theory $\mathbb{T}$</th>
<th>$\mathcal{C}$ a category</th>
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<tbody>
<tr>
<td>contexts $\Gamma, \Delta, \ldots$</td>
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<td>( A[\sigma] )</td>
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<td>pullback of ( \Gamma.A \twoheadrightarrow \Gamma ) along ( \Delta \xrightarrow{\sigma} \Gamma )</td>
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\[
\begin{align*}
\Delta.A[\sigma] & \xrightarrow{\sigma.A} \Gamma.A \\
\downarrow & \\
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Further, choice of pullbacks is functorial.

$$
\begin{array}{c}
\Delta \cdot A[\sigma] \xrightarrow{\sigma \cdot A} \Gamma \cdot A \\
\downarrow \quad \downarrow \\
\Delta \xrightarrow{\sigma} \Gamma
\end{array}
$$
Two ways to keep track of type dependencies ($\Gamma.A \rightarrow \Gamma$):
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**Contextual categories** have an $\mathbb{N}$-grading on objects

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\text{Ob}_0 \mathcal{C} \leftarrow \cdots \leftarrow \text{Ob}_n \mathcal{C} \leftarrow^{p_n} \text{Ob}_{n+1} \mathcal{C} \leftarrow \cdots
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such that $p_n(\Gamma.A) = \Gamma$ (where $\Gamma \in \text{Ob}_n \mathcal{C}$).
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**Categories with Attributes** have a functor

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such that $A \in \text{Ty}(\Gamma)$. 

Encoding type dependencies

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Homotopy-theoretic models

Ctx \rightarrow CwA \rightarrow Comp \leftarrow Trb
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Fibrations determine path objects, which can then be used to define homotopies $f \sim g$ and homotopy equivalences $X \simto Y$. 
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Tribes are categories with weak equivs. $:= \text{homotopy equivs}$. 
Contextual categories are tribes:

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Contextual categories vs. Tribes

Contextual categories are tribes:

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However, tribes are not contextual categories:

- no functorial choice of pullbacks
- no pullback-stable choice of $\text{Id}$- and $\Sigma$-types.
Comprehension categories encode pullbacks against $\rightarrow$ via a Grothendieck fibration $P : \mathcal{T} \rightarrow \mathcal{C}$.

**Logical Framework (LF):**
- Existence of categorical dependent exponentials.
- Satisfied if $\mathcal{C}$ has $\Pi$-types or is locally cartesian closed.
Comprehension categories encode pullbacks against $\to$ via a Grothendieck fibration $P: \mathcal{T} \to \mathcal{C}$.

They provide the right setting for strictifying these pullbacks:

**Theorem (Lumsdaine-Warren 2015)**

Let $\mathcal{C}$ be a full comprehension category satisfying LF, with weakly stable $\text{Id}$ and $\Sigma$-types.

There is an equivalent full split comprehension category $\mathcal{C}_1$ with strictly stable $\text{Id}$ and $\Sigma$-types.
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Comprehension categories encode pullbacks against \( \rightarrow \) via a Grothendieck fibration \( P : T \rightarrow C \).

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Let \( C \) be a full comprehension category satisfying LF, with weakly stable \( \text{Id} \) and \( \Sigma \)-types. ▼ ▼ ▼ tribes

There is an equivalent full split comprehension category \( C_1 \) with strictly stable \( \text{Id} \) and \( \Sigma \)-types. ▼ ▼ ▼ contextual cats/CwAs

**Logical Framework (LF):**
- Existence of categorical dependent exponentials.
- Satisfied if \( C \) has \( \Pi \)-types or is locally cartesian closed.
Need units and counits to be natural weak equivalences.
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Weak equivalences within each model
Given Id-types/path objects, we may define homotopies $f \sim g$ and homotopy equivalences $X \rightsquigarrow Y$. 

**Note**: LE assumes homotopies factor through chosen Id-types, and requires knowledge of ‘immediate’ type-dependencies.
Given Id-types/path objects, we may define homotopies \( f \sim g \) and homotopy equivalences \( X \simto Y \).

\( F : C \rightarrow D \) is a **logical equivalence** (LE) if types and terms in \( D \) have preimages in \( C \), up to homotopy (*weak type* and *term lifting*).

\( F : C \rightarrow D \) is a **homotopy-theoretic equivalence** (HE) if it induces an equivalence \( \text{Ho} C \cong \text{Ho} D \).
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We may generalize LE in two ways, with a view towards HE:

- Require homotopies/homotopy equivalences to factor through some Id-type (rather than a chosen Id-type)
- Require weak context and section lifting
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- Require homotopies/homotopy equivalences to factor through *some* Id-type (rather than a chosen Id-type)
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<td>Chosen Id-type</td>
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</tr>
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(Assume $\text{Id}, \Sigma, \text{LF}$)
Thank you!

Questions/comments?
- **Contexts** are finite lists \([x_1 : A_1, \ldots, x_n : A_n]\), up to definitional equality and renaming of variables

- **Substitutions** \([x_1 : A_1, \ldots, x_n : A_n] \xrightarrow{f} [y_1 : B_1, \ldots, y_m : B_m]\) are sequences

\[
\begin{align*}
x_1 : A_1, \ldots, x_n : A_n \vdash f_1 : B_1 \\
\vdots \\
x_1 : A_1, \ldots, x_n : A_n \vdash f_m : B_m
\end{align*}
\]
Definition (Cartmell 1978)

A **contextual category** is a category $\mathcal{C}$ with:

1. A grading on objects $\text{Ob}\, \mathcal{C} = \biguplus_{n \in \mathbb{N}} \text{Ob}_n\mathcal{C}$
2. A terminal object 1 which is the unique object in $\text{Ob}_0\mathcal{C}$
3. Maps $\text{pt}: \text{Ob}_{n+1}\mathcal{C} \to \text{Ob}_n\mathcal{C}$
4. *Canonical projections* $A \to \text{pt}A$
5. A functorial choice of pullbacks against canonical projections:

\[
\begin{align*}
A[\sigma] \xrightarrow{\sigma.A} A \\
\downarrow \quad \quad \quad \downarrow \\
\Delta \xrightarrow{\sigma} \text{pt}A
\end{align*}
\]

---

A **category with attributes** is a category $\mathcal{C}$ with:

1. A terminal object $1$
2. A functor $\text{Ty}: \mathcal{C}^{\text{op}} \to \text{Set}$
3. For each $A \in \text{Ty}(\Gamma)$, an object $\Gamma.A \in \mathcal{C}$ and map $\Gamma.A \to \Gamma$
4. A functorial choice of a map $\sigma.A$, for each $A \in \text{Ty}(\Gamma)$ and $\Delta \xrightarrow{\sigma} \Gamma$, fitting into a pullback square:

\[
\Delta.(\text{Ty}f)(A) \xrightarrow{\sigma.A} \Gamma.A
\]

\[
\Delta \xrightarrow{\sigma} \Gamma
\]
Definition (Joyal 2014, Shulman 2015)

A tribe is a category $\mathcal{C}$ with a subcategory $\mathcal{F}$ of fibrations ($\rightarrow$) containing all isomorphisms, such that

1. $\mathcal{C}$ has a terminal object, and all objects are fibrant
2. Pullbacks along fibrations exist, and fibrations are stable under pullback
3. Anodyne maps ($\sim \rightarrow$) are stable under pullback along fibrations
4. Every map factors as $\cdot \sim \rightarrow \cdot \rightarrow$.

Anodyne maps are those with the left lifting property against fibrations.

André Joyal. ‘Categorical homotopy type theory’. In: Slides from a talk at MIT dated 17 (2014).

## Paths and homotopies within tribes

### Definition

A **path object** for $Y \in \mathcal{C}$ is a factorization of the diagonal:

$$
\Delta_Y = (Y \sim P_Y \to Y \times Y)
$$

Two maps $f, g : X \to Y$ are **homotopic** ($f \sim g$) if there is a factorization of $(f, g)$ through $P_Y$:

$$(f, g) = (X \xrightarrow{H} P_Y \to Y \times Y)$$

A map $f : X \to Y$ is a **homotopy equivalence** ($f : X \sim Y$) if there is a map $g : Y \to X$ such that $gf \sim 1_X$ and $fg \sim 1_Y$. 

---

$\hat{\text{every\ tribe}}$ is a category with weak equivalences, where $\text{Weak\ equivalences} \equiv \text{Homotopy\ equivalences}$
Paths and homotopies within tribes

**Definition**

A **path object** for $Y \in \mathcal{C}$ is a factorization of the diagonal:

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A map $f : X \to Y$ is a **homotopy equivalence** ($f : X \xrightarrow{\sim} Y$) if there is a map $g : Y \to X$ such that $gf \sim 1_X$ and $fg \sim 1_Y$.

Every tribe is a category with weak equivalences, where

**Weak equivalences** := Homotopy equivalences
A **comprehension category** consists of a Grothendieck fibration \( T \xrightarrow{P} C \) and a **comprehension functor** \( \chi : T \to \text{Arr} C \) such that the diagram
\[
\begin{array}{ccc}
T & \xrightarrow{\chi} & \text{Arr} C \\
\downarrow P & & \downarrow \text{cod} \\
C & & 
\end{array}
\]
commutes, and \( \chi \) sends cartesian maps to pullback squares.
A comprehension category is **split** if \( P \) is a split fibration, and **full** if \( \chi \) is fully faithful.

Categories with attributes are full split comprehension categories.

---

Let $\mathcal{C}$ be a model of type theory, with Id-types which give rise to homotopies $f \sim g$. 
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**Definition**

The **homotopy category** of $\mathcal{C}$ is the category $\text{Ho}\mathcal{C}$ with the same objects, and homotopy classes of maps.

**Definition**

A functor of models $F: \mathcal{C} \to \mathcal{D}$ is a **homotopy equivalence** (HE) if it induces an equivalences of categories $\text{Ho}\mathcal{C} \cong \text{Ho}\mathcal{D}$. 
Logical equivalence

Definition

A functor $F : C \rightarrow D$ is a **logical equivalence** (LE) if it satisfies:

1. **Weak type lifting**: For all $\Gamma \in C$, $A \in \text{Ty}(F \Gamma)$, there exists $\tilde{A} \in \text{Ty}(\Gamma)$ and $F \tilde{A} \sim A$ over $F \Gamma$.

   ![Diagram of weak type lifting]

2. **Weak term lifting**: For all $\Gamma \in C$, $A \in \text{Ty}(\Gamma)$ and $a \in \text{Tm}(FA)$, there exists a term $\tilde{a} \in \text{Tm}(A)$ such that $F \tilde{a} \sim a$.

   ![Diagram of weak term lifting]
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   ![Diagram of weak type lifting](https://via.placeholder.com/150)

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\[
\begin{array}{c}
\tilde{A} \\
\downarrow \\
\Gamma
\end{array} \quad \Rightarrow 
\begin{array}{c}
F \tilde{A} \\
\downarrow \\
F \Gamma
\end{array} \quad \sim 
\begin{array}{c}
A \\
\downarrow \\
\Gamma
\end{array}
\]

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\downarrow \\
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FA \\
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   ![Diagram of weak type lifting](image)

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   ![Diagram of weak term lifting](image)
A functor $F : \mathcal{C} \to \mathcal{D}$ is a **logical equivalence** (LE) if it satisfies:

1. **Weak type lifting**: For all $\Gamma \in \mathcal{C}$, $A \in \text{Ty}(F\Gamma)$, there exists $\tilde{A} \in \text{Ty}(\Gamma)$ and $F\tilde{A} \sim A$ over $F\Gamma$.

$$
\begin{array}{c}
\tilde{A} \\
\downarrow \\
\Gamma
\end{array} \quad \Rightarrow \quad 
\begin{array}{c}
F\tilde{A} \\
\downarrow \\
F\Gamma
\end{array} \quad \sim \quad 
\begin{array}{c}
A \\
\uparrow \\
a
\end{array}
$$

2. **Weak term lifting**: For all $\Gamma \in \mathcal{C}$, $A \in \text{Ty}(\Gamma)$ and $a \in \text{Tm}(FA)$, there exists a term $\tilde{a} \in \text{Tm}(A)$ such that $F\tilde{a} \sim a$.

$$
\begin{array}{c}
A \\
\uparrow \\
\tilde{a}
\end{array} \quad \Rightarrow \quad 
\begin{array}{c}
FA \\
\downarrow \\
a
\end{array} 
\begin{array}{c}
\Gamma \\
\uparrow \\
\downarrow \\
\tilde{a} \\
\downarrow \\
\Gamma
\end{array} \quad \sim \quad 
\begin{array}{c}
F\Gamma
\end{array}
$$
A functor $F : \mathcal{C} \to \mathcal{D}$ is a logical equivalence (LE) if it satisfies:

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   ![Diagram of weak type lifting]

2. **weak term lifting**: For all $\Gamma \in \mathcal{C}$, $A \in \text{Ty}(\Gamma)$ and $a \in \text{Tm}(FA)$, there exists a term $\tilde{a} \in \text{Tm}(A)$ such that $F \tilde{a} \sim a$.

   ![Diagram of weak term lifting]
Logical equivalence

**Definition**

A functor $F: \mathcal{C} \to \mathcal{D}$ is a **logical equivalence** (LE) if it satisfies:

1. **weak type lifting**: For all $\Gamma \in \mathcal{C}$, $A \in \text{Ty}(F\Gamma)$, there exists $\tilde{A} \in \text{Ty}(\Gamma)$ and $F\tilde{A} \sim A$ over $F\Gamma$.

\[
\begin{array}{ccc}
\tilde{A} & \longrightarrow & F\tilde{A} \\
\downarrow & & \downarrow \sim \\
\Gamma & \longrightarrow & F\Gamma
\end{array}
\]

2. **weak term lifting**: For all $\Gamma \in \mathcal{C}$, $A \in \text{Ty}(\Gamma)$ and $a \in \text{Tm}(FA)$, there exists a term $\tilde{a} \in \text{Tm}(A)$ such that $F\tilde{a} \sim a$.

\[
\begin{array}{ccc}
A & \longrightarrow & FA \\
\tilde{a} & \uparrow & \sim \\
\Gamma & \longrightarrow & F\Gamma
\end{array}
\]