The Enriched Grothendieck Construction

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In the unenriched world, things are often obscured by all kinds of coincidences.

- *this talk*

Objectives

Main objective: Present an enriched version of

Theorem (Grothendieck, 1964)

Let B be a category. There is a 2-equivalence

$$\mathbf{Fib}(B) \cong 2\text{-}\mathbf{Fun}(B^{op}, \mathbf{Cat})$$

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Secondary objective: Highlight things we take for granted in the unenriched setting (i.e. when enriching over **Set**)

Outline

1. Enriched category theory
2. Fibrations
3. Results
4. $\mathcal{V} = \mathbf{Vect}_k$?

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Enriched categories

Capture the idea of a category whose homs $\mathcal{C}(x, y)$ have extra structure i.e. belong to a monoidal category $(\mathcal{V}, \otimes, \mathbf{1})$.

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$\otimes$	$\mathbf{1}$	$\mathcal{V}$	$\mathcal{V}$-categories
$\times$	$*$	**Set**	categories
		Cat	strict 2-categories
		sSet	simplicial categories
$\otimes_k$	k	$\mathbf{Vect}_k$	k-linear categories
		$\mathbf{Ch}_R$	differential graded categories

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Also, monoids in $\mathcal{V}$ are $\mathcal{V}$-categories with one object:

$$\mathrm{Ob}(\mathcal{C}) = \{*\} \qquad \mathcal{C}(*, *) = G, \text{ a monoid}$$

Underlying categories and free $\mathcal{V}$-categories

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Conversely, there is often a *free* $\mathcal{V}$*-category* $C_{\mathcal{V}}$ on a category C.

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One-object example when $\mathcal{V} = \mathbf{Vect}_k$:

C	$C_{\mathcal{V}}$	$(C_{\mathcal{V}})_0$
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Note: $C \neq (C_{\mathcal{V}})_0$, but we do have $C \hookrightarrow (C_{\mathcal{V}})_0$.

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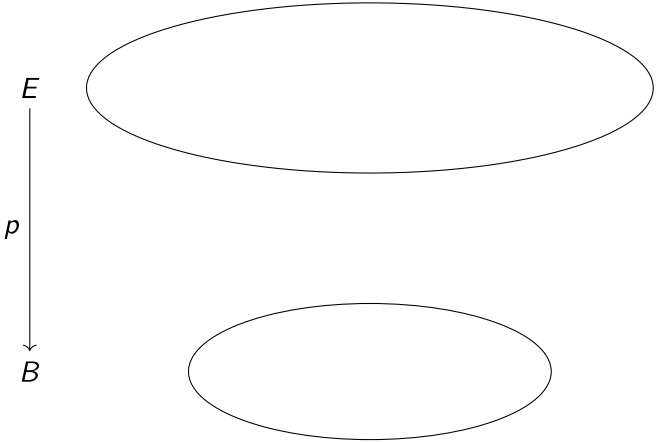

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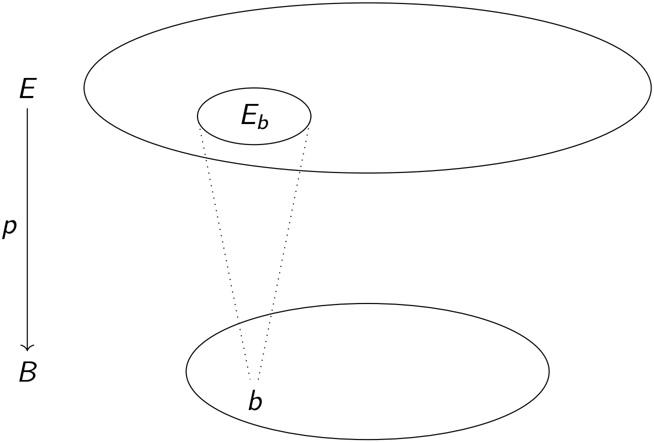

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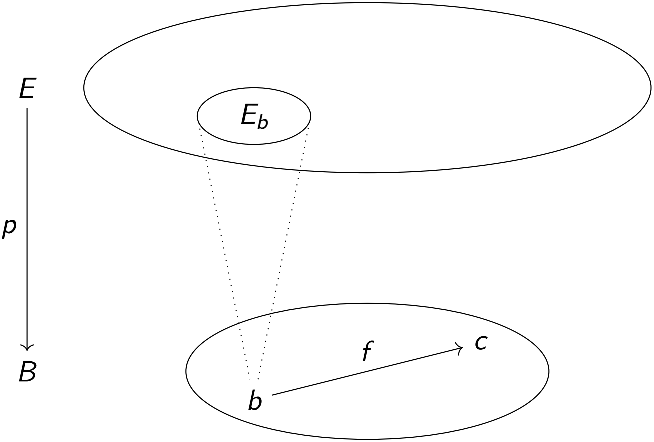

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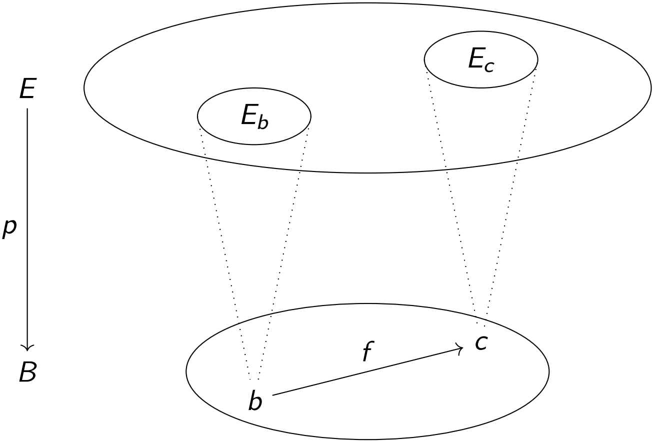

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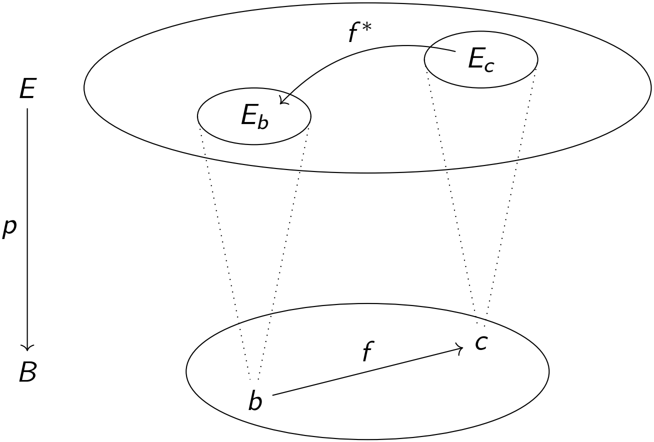

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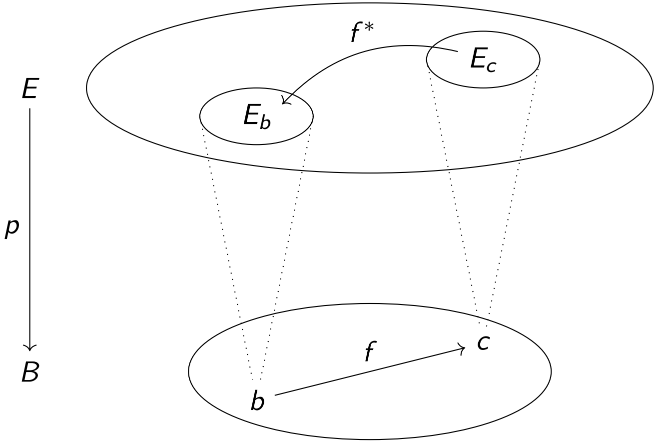

Further, morphisms in E are determined by those in the fibers.

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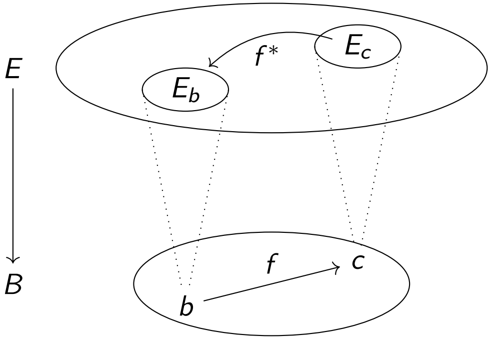

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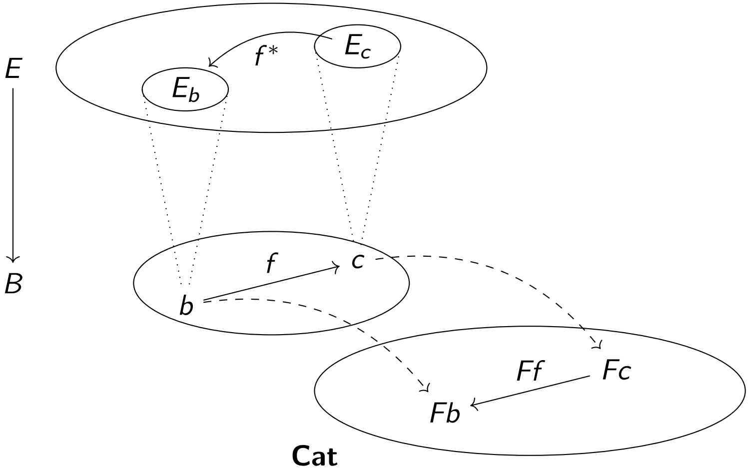

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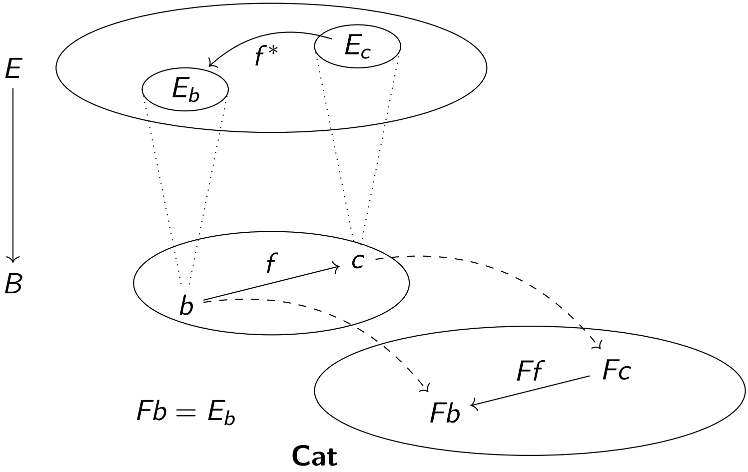

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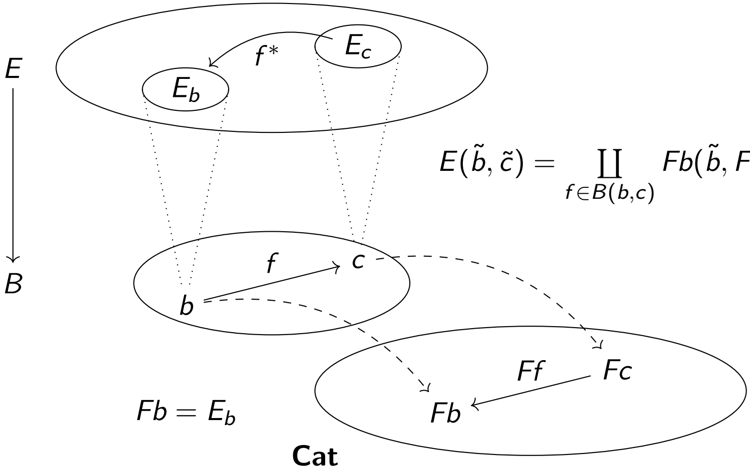

$$E(\tilde{b}, \tilde{c}) = \coprod_{f \in B(b,c)} Fb(\tilde{b}, Ff\, \tilde{c})$$

Outline

Results

Proposition (Beardsley-W.)

Let $\mathcal{B}$ be a $\mathcal{V}$-category. There is a 2-functor

$$\mathcal{V}\text{-}\mathbf{Fib}(\mathcal{B}) \to 2\text{-}\mathbf{Fun}({\mathcal{B}_0}^{op}, \mathcal{V}\text{-}\mathbf{Cat}).$$

Proposition (Beardsley-W.)

Suppose the unit $\mathbf{1} \in \mathcal{V}$ is terminal, and pullbacks commute with coproducts in $\mathcal{V}$. Let B be a category. There is 2-functor

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Need to precompose with $B \hookrightarrow (B_{\mathcal{V}})_0$.

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When $\mathcal{V} = \mathbf{sSet}$ [Lurie 2009] or $\mathbf{Cat}$ [Buckley 2014], there are enhanced equivalences that work for arbitrary $\mathcal{V}$-categories $\mathcal{B}$.

Outline

Problems with $\mathcal{V} = \mathbf{Vect}_k$

- The unit k is not terminal
- Pullbacks don't commute with coproducts
- Even binary products don't commute with coproducts (both are $\oplus$):

$$X \times (Y \amalg Z) \neq (X \times Y) \amalg (X \times Z)$$
$$X \oplus (Y \oplus Z) \neq (X \oplus Y) \oplus (X \oplus Z)$$

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But $\otimes_k$ does commute with $\oplus$:

$$X \otimes (Y \oplus Z) = (X \otimes Y) \oplus (X \otimes Z)$$

Is there a related category in which $\otimes$ is a product, k is terminal, and pullbacks commute with coproducts?

Coalgebras

The category $\mathbf{Coalg}_k = \mathbf{Comon}(\mathbf{Vect}_k)$:

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But maybe this is the wrong question.

Another perspective: comonoids and coactions

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Theorem (Cohen & Montgomery 1984,. . . ,Tamaki 2009)

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Are fibrations actually special 'coaction functors'?
What other results are secretly about comonoids/coactions?

Thank you!

Questions/comments?