The Grothendieck Construction for Enriched, Internal and ∞-Categories

Liang Ze Wong

Final Exam

26 Feb 2019

Publications

BW1 Jonathan Beardsley and Liang Ze Wong. *The enriched Grothendieck construction*. Advances in Math, 2019.

BW2 __________. *The operadic nerve, relative nerve, and the Grothendieck construction*. arXiv:1808.08020, 2018.

W Liang Ze Wong. *Smash products for Non-cartesian Internal Prestacks*, 2019.

- Alex Chirvasitu, S Paul Smith and Liang Ze Wong. *Noncommutative geometry of homogenized quantum* $\mathfrak{sl}(2,\mathbb{C})$, Pacific Journal of Math, 2017.
- Krzysztof Kapulkin, Zachery Lindsey and Liang Ze Wong. *A co-reflection of cubical sets into simplicial sets with applications to model structures*, 2019.
- Simon Cho, Cory Knapp, Clive Newstead and Liang Ze Wong. *Weak equivalences between categories of models of type theory*. (in preparation)

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Also have a *split* surjection:

$$N = \ker \pi \hookrightarrow N \rtimes G \underset{\longleftarrow}{\overset{\pi}{\twoheadrightarrow}} G$$

And we can recover N by taking the kernel of π.

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Splitting Lemma (Classical)

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Today, we'll see that G and N don't have to be groups:
They can be algebras, categories, ∞-categories, and more!

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i.e. a functor $N_\bullet \colon C \to \mathbf{Cat}$

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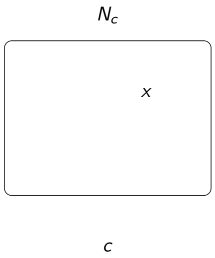

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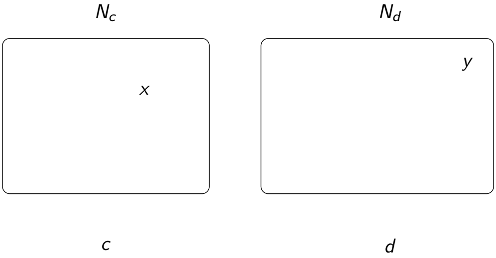

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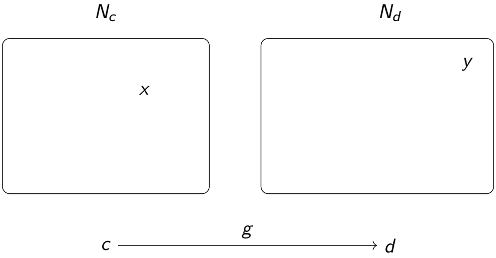

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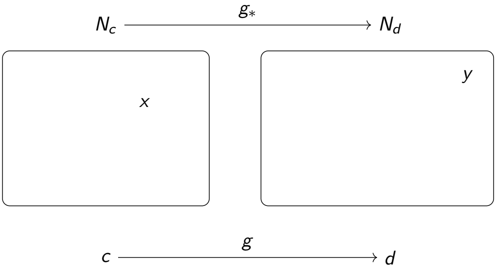

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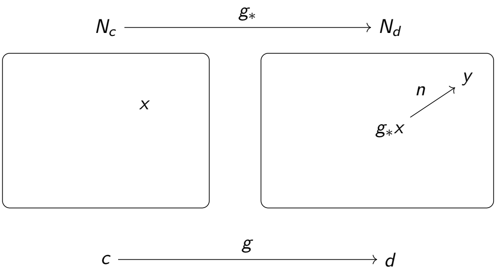

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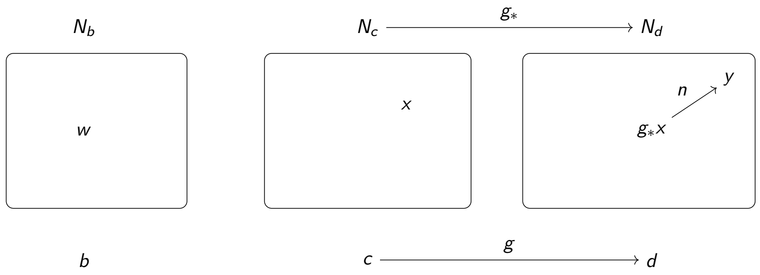

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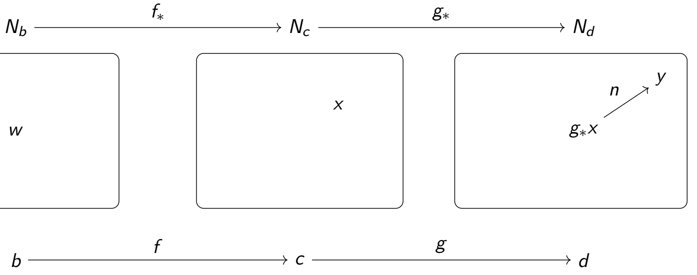

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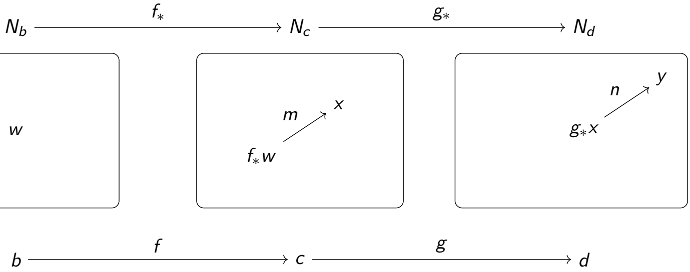

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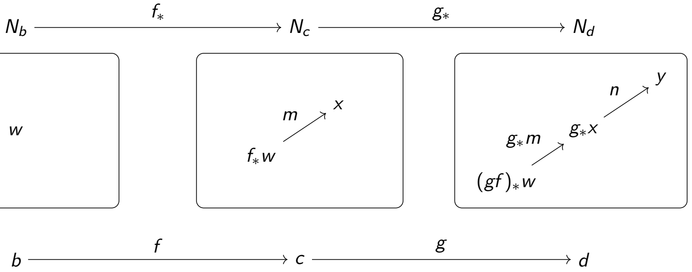

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Theorem (Grothendieck 1959)

There is an isomorphism of categories:

$$\left\{ \begin{array}{c} \textit{Functors} \\ N_\bullet : C \to \mathbf{Cat} \end{array} \right\} \underset{\text{fibers}}{\overset{\rtimes}{\rightleftarrows}} \cong \left\{ \begin{array}{c} \textit{Split opfibrations} \\ N_\bullet \rtimes C \to C \end{array} \right\}$$

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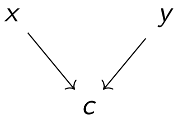

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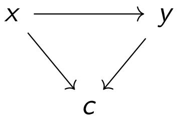

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$(C/\bullet) \rtimes C = \mathbf{Arr}\,C$ and $\mathbf{Arr}\,C \to C$ is the codomain functor.

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This is the *global module category* $\mathbf{Mod}$.

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But we don't have an algebra map $A \rtimes G \to kG \ldots$

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The coaction perspective allows us to replace kG with any bialgebra or Hopf algebra H.

The Skew Group Ring and Smash Products

Theorem (Cohen-Montgomery 1984)

For G a group, there is a bijective correspondence:

$$\left\{ \begin{array}{c} G\text{-actions} \\ G \times A \to A \end{array} \right\} \underset{\text{fibers}}{\overset{\rtimes}{\underset{\longleftarrow}{\overset{\longrightarrow}{\cong}}}} \left\{ \begin{array}{c} G\text{-graded algebras} \\ A \rtimes G \end{array} \right\}$$

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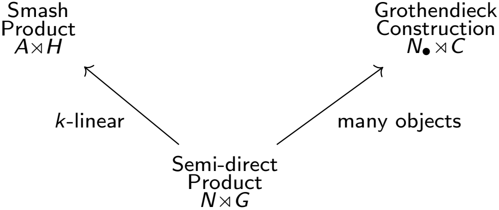
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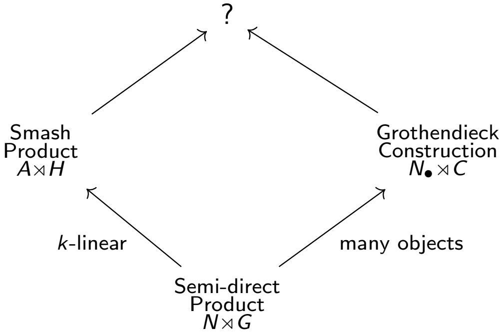
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Enriched and Internal Categories

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- a *set* of objects C_0
- for all $x, y \in C_0$, a *set* of arrows $\mathrm{Hom}_C(x, y)$

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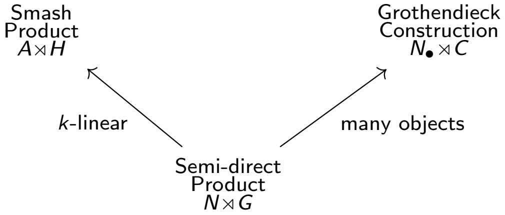
(possibly with other properties, e.g. cocommutativty)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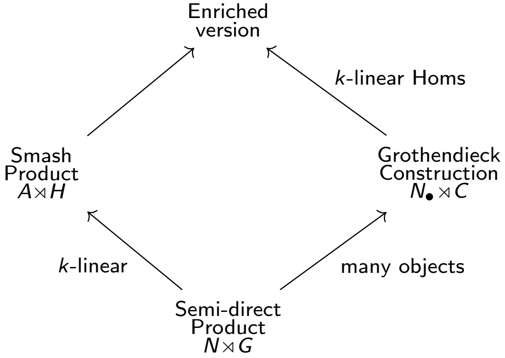
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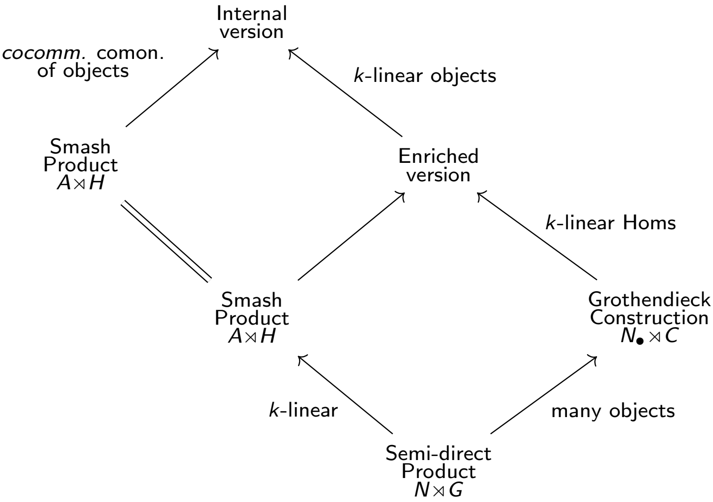
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Let $\mathcal{C}$ be a comonoidal $\mathcal{V}$-category. Then

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Theorem (W)

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Suppose $\mathcal{V}$ has equalizers, and $\otimes$ preserves them. Let $\mathcal{C}$ be a comonoidal internal category. Then

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When $\mathcal{C} = (k, H), \mathcal{A} = (k, A)$, this is just $A \boxtimes_k (H \boxtimes_k k) \cong A \otimes H$.

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$$A_1 \quad \boxtimes_{A_0} \quad C_1 \quad \boxtimes_{C_0} \quad A_0 \quad \boxtimes_{A_0} \quad A_1 \quad \boxtimes_{A_0} \quad C_1 \quad \boxtimes_{C_0} \quad A_0$$

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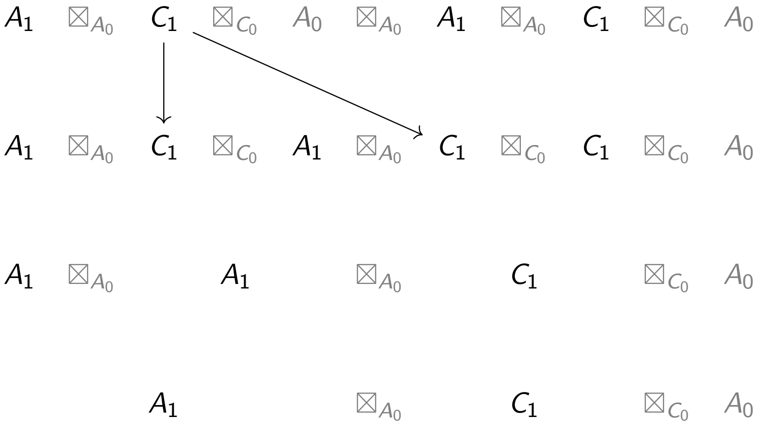

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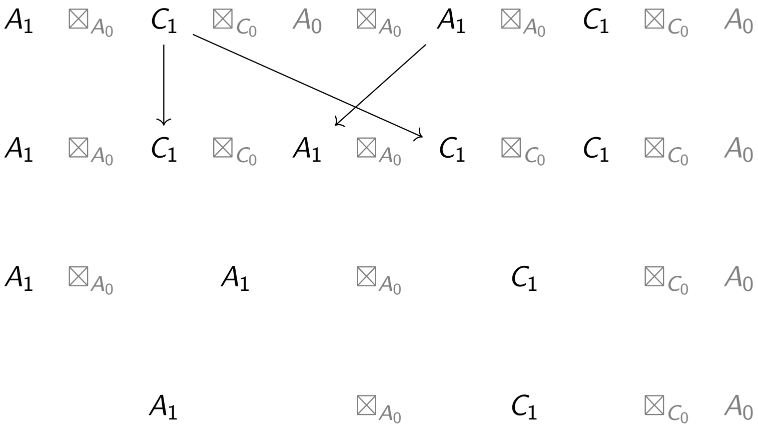

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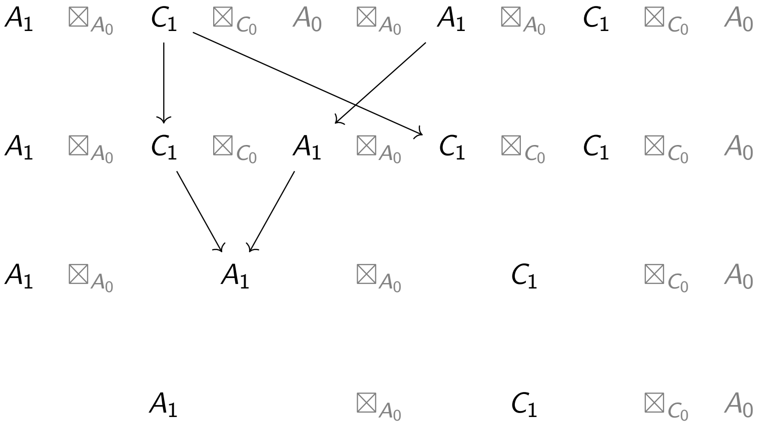

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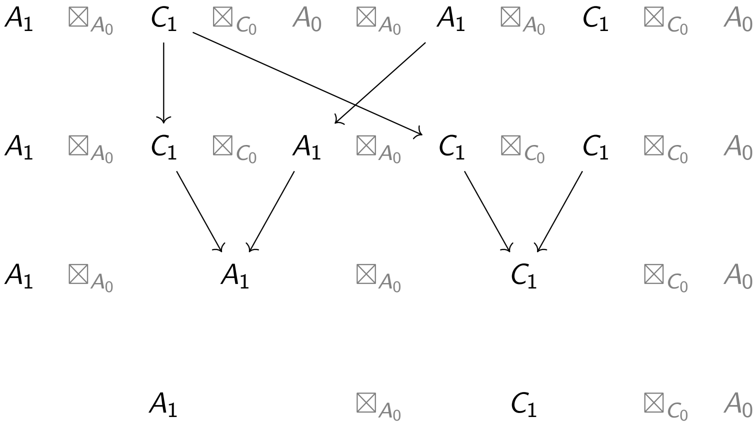

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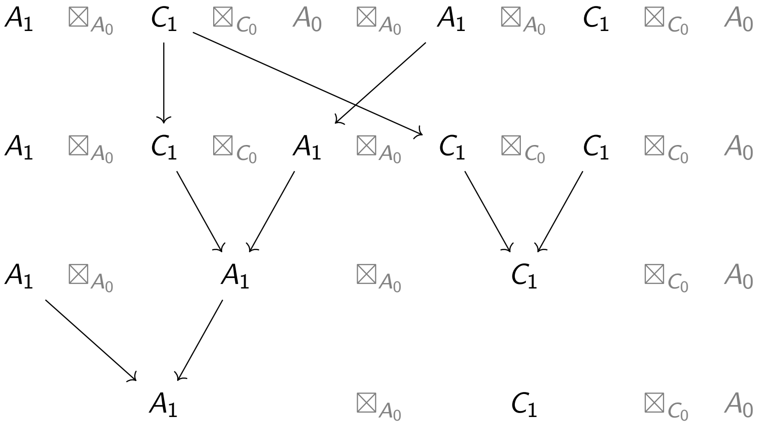

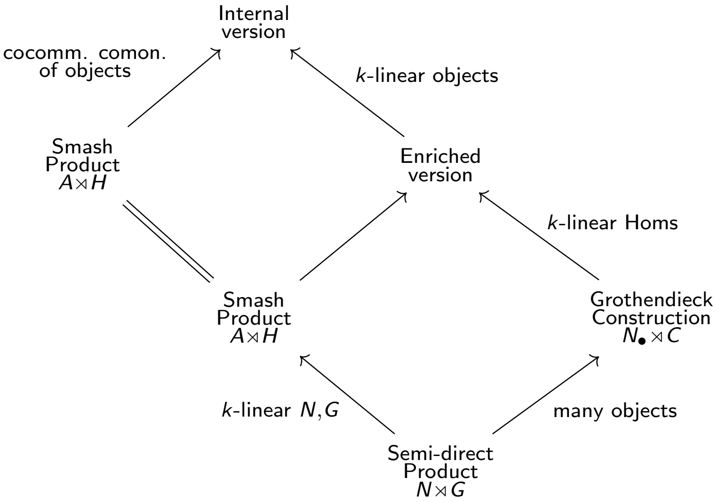
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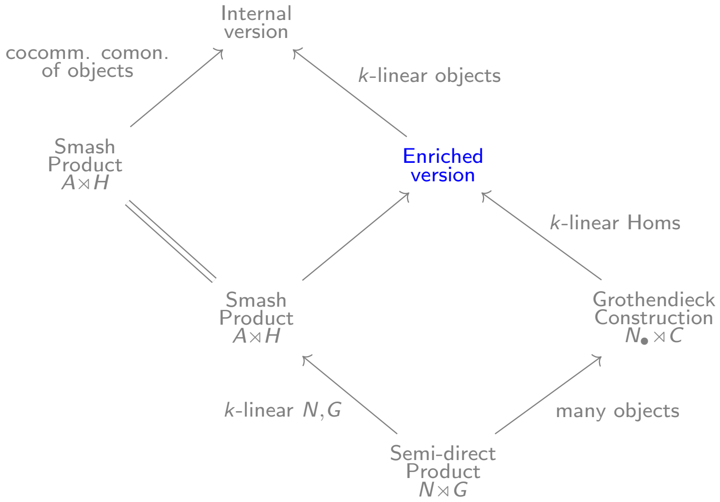

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Theorem (Cibils-Marcos 2006, Lowen 2008, Tamaki 2009)

Suppose $\mathcal{V}$ has coproducts, and $\otimes$ preserves them. Then:

$$\left\{ \begin{array}{c} \textit{Functors} \\ \mathcal{A}_\bullet \colon C \to \mathcal{V}\text{-}\mathbf{Cat} \end{array} \right\} \underset{\text{fibers}}{\overset{\rtimes}{\rightleftarrows}} \cong \left\{ \begin{array}{c} C\textit{-graded } \mathcal{V}\textit{-cats} \\ \mathcal{A}_\bullet \rtimes C \end{array} \right\}$$

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Suppose further that $\mathbf{1}$ is terminal, $\mathcal{V}$ has pullbacks, and pullbacks and $\mathrm{Hom}_{\mathcal{V}}(\mathbf{1}, -)$ *preserve coproducts.*

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e.g. $\mathcal{V} = \mathbf{sSet}$

Simplicial sets and ∞-categories

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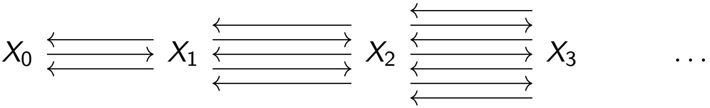

points paths homotopies

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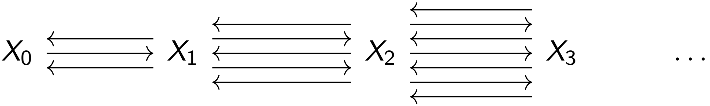

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Simplicial sets are thus combinatorial models of topological spaces.

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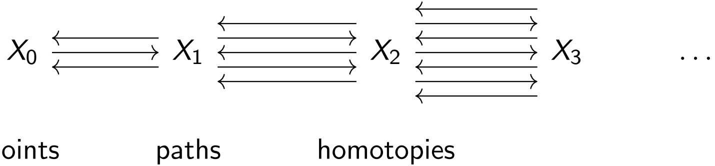

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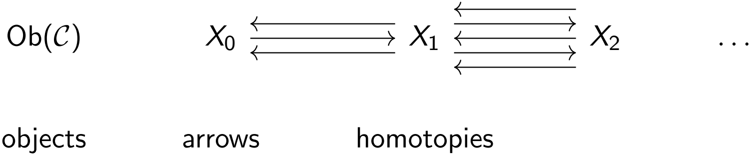

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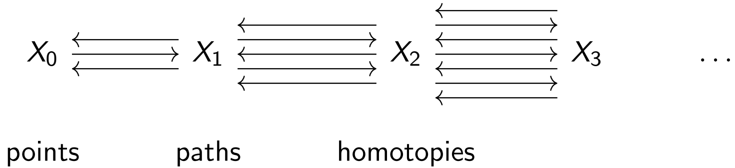

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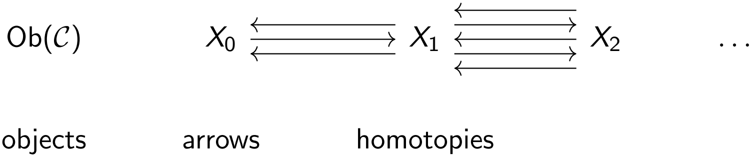
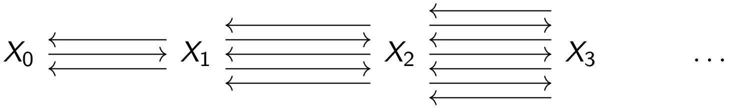


i.e. an ∞-category!

But simplicial sets themselves model ∞-categories:



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And both models are related:

$$\mathbf{sSet} \underset{\mathrm{N}}{\overset{\mathfrak{C}}{\underset{\perp}{\rightleftarrows}}} \mathbf{sCat}$$

∞-categorical Grothendieck construction

Have an ∞-categorical version in terms of (marked) simplicial sets:

Theorem (Lurie 2009)

$$\left\{ \begin{array}{c} \textit{Simplical maps} \\ A_\bullet : S \to \mathbf{Cat}_\infty \end{array} \right\} \underset{\simeq}{\overset{\rtimes}{\rightleftarrows}} \left\{ \begin{array}{c} \textit{Cocartesian fibrations} \\ A_\bullet \rtimes S \to S \end{array} \right\}$$

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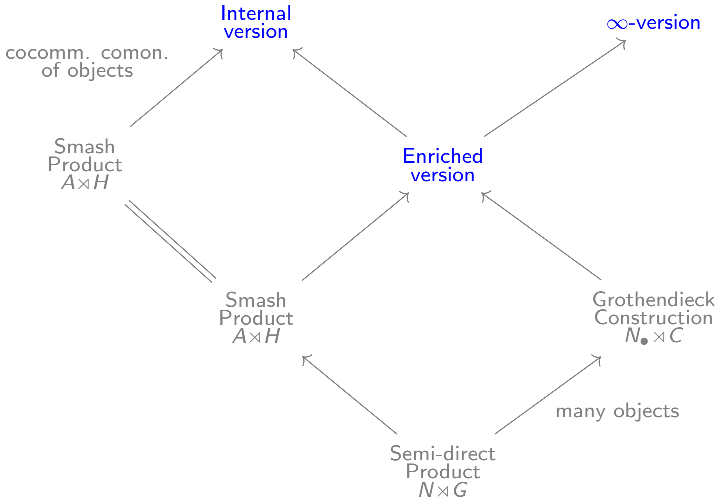
$$\mathrm{N}(A_\bullet) \rtimes \mathrm{N}(C) \cong \mathrm{N}\left(\mathcal{A}_\bullet \rtimes C\right).$$


Internal version
∞-version
cocomm. comon. of objects
Smash Product $A \rtimes H$
Enriched version
Smash Product $A \rtimes H$
Grothendieck Construction $N_\bullet \rtimes C$
many objects
Semi-direct Product $N \rtimes G$

Thank you!

Questions?

Application: Monoidal ∞-categories

Recall the *simplex category* Δ:

- objects are $[n] = \{0 \leq 1 \leq \cdots \leq n\}$
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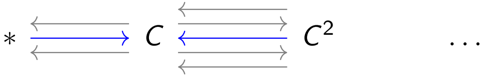

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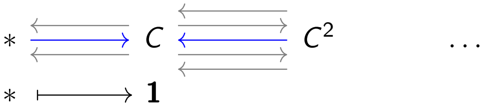

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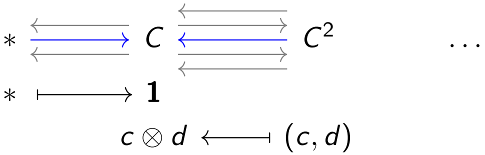

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$C^\otimes := C^\bullet \rtimes \Delta^{\mathrm{op}}$

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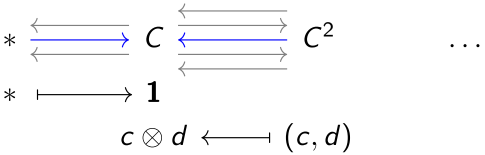

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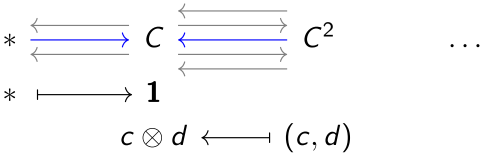

$C^{\otimes} := C^{\bullet} \rtimes \Delta^{\mathrm{op}}$ has an opfibration down to Δ^{op}. In fact, we can *define* monoidal categories in terms of opfibrations $M \to \Delta^{\mathrm{op}}$.

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Proposition (Lurie 2007)

A simplicial monoidal category $(C, \otimes, \mathbf{1})$ *gives rise to a monoidal* ∞*-category* $\mathrm{N}(C^{\otimes})$.

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This gives a better handle on coalgebras in monoidal ∞-categories arising from simplicial monoidal categories.